

¹ Nabeela Abu-Alkishik² Jamal Oudetallah³ Eman Almuhr⁴ Hamza Qoqazeh⁵ Maryam Alholi⁶ Majd jawarneh

On C-Compactness in Tri-Topological Spaces



Abstract: - This paper intends to define the C-compact spaces with three topologies called tri-C-compact space. As well as study properties along with relations with some other tritopological spaces. Several theoretical results are stated and proved through generalizing lots of well-known theorems about C-compact spaces. These results are supported via handling some illustrative examples.

Keywords: Tri C-Compactness, Tri C-Separation Axioms, Tri extremely disconnected space.

I. INTRODUCTION

At the beginning of our introduction, it is necessary to shed light on the concept of bitopological spaces and an overview of the research and studies on them, which are considered the starting point and basis for studies related to triple topological spaces. Indeed, most of the research on tri- topological spaces came as a generalization of binary studies and clarifies the relationship between them.

The basic concepts of bitopological spaces go back to Kelly [1]. In 1963, he defined the bitopological space as a non-empty set ξ defined on it two topologies and a symbol for this space $(\xi, \wp_1, \wp_2)$. During this study, many spaces were defined, such as pairwise regular and pairwise normal spaces. He further generalized several topological concepts, that includes Hausdörff and pairwise Hausdörff. This approach captured the concern of researchers, leading to the definition of pairwise compact spaces and extensive studies on their properties.

Since that time, most researchers interested in this field have used these concepts to define many bitopological concepts, such as separation axioms, covering properties, and the relationships between these concepts. For more on that, we mention these studies: Fletcher.et.al [26] and kim [2].

Knowing that these studies focused on defining and studying the properties of compact binary spaces and discussing examples for each case.

In 1969, the scientists Fletscher. et.al [26], defined a number of types of covers in bi topological spaces, such as p –open and $\tau_1\tau_2$ –open covers. A cover $\tilde{U} = \{u_\alpha: \alpha \in \Delta\}$ of a bitopological space $(\xi, \wp_1, \wp_2)$. is called $\tau_1\tau_2$ –open cover if $\tilde{U} \subseteq \wp_1 \cup \wp_2$. If $\tilde{U}$ contains at least one member of both $\wp_1$ and $\wp_2$ then $\tilde{U}$ is called p –open cover.

Through it, each of the following spaces was defined such as pairwise compact, pairwise countably compact spaces and others.

This study is considered one of the best studies that contributed to establishing research work in binary topological spaces.

In 1972, semi compactness of bitopological spaces were defined by the scientist Datta [27]. Also the properties of this concept were discussed. See also Bersan [4].

In 1975, Scientists Cook and Reilly [2] discussed the relationship between the following topological concepts: p –compact, s –compact and B –compact spaces.

In continuation of the distinguished scientific approach to this subject, many important studies have come in this field. We mention, for example, the Fora and Hdieb study [9] in 1983, which gave an adequate generalization for most of the previous studies, and through it the following concepts were presented: pairwise Lindelöf, semi Lindelöf and B - Lindelöf spaces. Their findings contributed to the development and validation of numerous

¹ Mathematics, Faculty of Science, Jerash University, Jerash, JORDAN. nabeelakishik@yahoo.com

² Mathematics, University of Petra, Amman, Jordan. jamalayasrah12@gmail.com

³ Mathematics, Faculty of Science, Applied Science Private University, Amman, JORDAN. e_almuhr@asu.edu.jo

⁴ Mathematics, Irbid National University, Irbid, Jordan. hhaqq983@gmail.com

⁵ Applied, Taibah University Al Ula, Saudi Arabia. Mholi@taibahu.edu.sa

⁶ Mathematics, Irbid National University, Irbid, Jordan. majdraed1997@yahoo.com

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topological concepts in the field. The relationship between these concepts and other well-known topological concepts is described. It is worth noting that this study shed light on the definition of important types of functions defined on bitopological spaces, including:

A function $h: (\xi, \wp_1, \wp_2) \rightarrow (\xi, \wp'_1, \wp'_2)$ is called P -open (P -closed, P -continuous, P -homeomorphism, respectively), if both functions

$h: (\xi, \wp_1) \rightarrow (\xi, \wp'_1)$ and $h: (\xi, \wp_2) \rightarrow (\xi, \wp'_2)$ are open (closed, continuous, homeomorphism, respectively).

In 2000, Dochviri [24], generalized the notions of semi-compact and Semi-closed for bitopological spaces. The characteristics of (i, j) -Semi-compact and (i, j) -quasi-H-Campart are given.

In 2007, Al-Hawary and Al-Omari [25] introduced the notion of b -open sets and use it to define b -continuous functions and pairwise b -continuous function.

We now begin our conversation about the main topic of the study:

The notion of a C -compact in topological space (X, τ) was introduced by [Arthur, 1945]. Further intensive study of such spaces has been done since then see for examples ([Engleking, 1989], [Willard, 1970], [Matveev, 1994]). In this paper, we study the notion of tri - C -compact, and tripartite locally C -compact in Tritopological spaces and derive some related results.

The study of tritopological spaces generalizes bitopological spaces, which themselves generalize single topological spaces. This research focuses on a non-empty set ξ equipped with three topologies $\wp_1, \wp_2, \wp_3$, referred to as a tritopological space, denoted by $(\xi, \wp_1, \wp_2, \wp_3)$.

Inter relationships, as seen in works [Atoom. et. al, 2021]. This summary outlines the research, concepts, and studies in bitopological spaces that laid the groundwork for future generalizations in tritopological spaces and beyond, which will be introduced in our study.

The idea of scientific research in tri-topological spaces has attracted the attention of many scientists, and most of these works have focused on generalizing the known concepts and ideas in topological spaces.

The exploration of tritopological spaces began in 2000 with [Kover, 2000], where he defined important separation axioms and clarified through various distinctive examples. These studies inspired numerous researchers to delve deeper into this field. In 2011, new separation axioms were introduced for tritopological spaces, based on open sets, namely 123-T0, 123-T1 and 123-T2 spaces, as detailed in [Hameed and Abid, 2011].

These concepts led to the definition of certain separation axioms in tri-spaces that are based on types of open sets specific to tritopological spaces, rather than traditional open sets. Three new separation axioms were introduced, referred to as 123b-T0, 123b-T1 and 123b-T2 spaces, as noted in [Hameed and Abid, 2011]. Additionally, in 2011, the doctoral dissertation by [Palaniammal, 2011] provided an in-depth study of tritopological spaces, resulting in significant findings, including the development of definitions for tri- α -continuous and tri- β -continuous functions. Sweedy and Hassan, 2011, introduced the concept of δ -continuous functions, detailing their properties in tritopological spaces. the topic of α -continuous functions in tri-spaces was studied, with a statement of their properties and their effect on tri-topological spaces. Following this, in 2016 a study by [Tapi, et. al. 2016] focused on presenting two open sets in tritopological spaces: semi-open sets and pre-open sets, exploring their fundamental properties. It also examined two important concepts of continuous functions: tri-semi-continuous and tri-pri-continuous functions, along with their associated properties. They also gave the concepts of a α - T -open sets with its properties. They use this to define one of an important types of function called. α - T -functions.

The explanation of bi-topological space's expandability, almost expandability, and feebly expandability was provided by (Oudetallah, 2018) and (Oudetallah, 2021). This chapter's major goal is to introduce and examine tripartite c -compact space, a novel kind of tripartite compact space. In 2023 [Hamzah et.al] they conducted a new and comprehensive study on tritopological spaces, focusing on generalizing well-known results from single and pairwise Lindelöf spaces. This study led to the establishment of new theories, results, and generalizations in tritopological spaces.

In section one, we study the concept of tri - C -compactness in tri -topological spaces introduce some properties, and relate it to other spaces. We study a well-known definition that will be used in the sequel. In section two, we study the concept of tripartite locally C -compactness in tritopological spaces, and prove several properties of these spaces. The terms τ_u , τ_{dis} , τ_{cof} , and τ_{coc} will denote the usual, discrete, cofinite, and the co -countable topologies, respectively.

Definition 1.1. [17] A cover U of tri -topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is called $\wp_1\wp_2\wp_3$ -open if $U \subset (\wp_1 \cup \wp_2 \cup \wp_3)$. If U contains at least one member of $\wp_i: i = 1, 2, 3$, then U is called $\wp_{1,2,3}$ -open cover.

Definition 1.2. [17] 1. A tri -topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is S -compact if for any $\wp_1\wp_2\wp_3$ -open cover of the space $(\xi, \wp_1, \wp_2, \wp_3)$ has a finite subcover.

2. A tri-topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is T -compact if any $\wp_{1,2,3}$ -open cover of the space $(\xi, \wp_1, \wp_2, \wp_3)$ has a finite subcover.
3. A tri-topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is C -compact if any $\wp_i$ -open cover of $\xi, i = 1, 2, 3$ has $\wp_j$ -finite subcover, $j = 1, 2, 3$ and $\wp_k$ -finite subcover, where $i \neq j \neq k$.

II. SOME DEFINITION IN Tri-TOPOLOGICAL SPACE

2.1 On tri - c -compact space

We define the tri - c -compact concept in a triple topological space.

Definition 2.1. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ be a tri -topological space, and $z \subset \xi$, then $\forall i = 1, 2, 3$:

- (i) z is tri-regular open if $z = \text{Int}_{\wp_i}(CL(z))$.
- (ii) z is tri-regular closed if $z = CL_{\wp_i}(\text{Int}(z))$.
- (iii) z is tri-semi-open if $\exists q_{\wp_i}$ an open set: $q_{\wp_i} \subseteq z \subseteq CL_{\wp_i}(q)$.

Remark 2.2. If $(\xi, \wp_1, \wp_2, \wp_3)$ is a tri -topological space and $z \subset \xi$, then:

- (i) If $\xi - z$ is tri -regular open, then z is tri -regular closed.
- (ii) If $\xi - z$ is tri -regular closed, then z is tri -regular open.

Theorem 2.3. If $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri-topological space, then each tri-open set is tri-semi-open.

Proof. If z is a tri - regular open set. Then, $\text{Int}_{\wp_i}(z) \subseteq z \subseteq CL_{\wp_i}(z) \forall i = 1, 2, 3$.

So, $q_{\wp_i} \subseteq z \subseteq CL_{\wp_i}(q)$. Therefore, z is a tri-semi-open set.

Definition 2.4. [16] If all open cover of $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ has a finite subcollection where its closure covers ξ , then $(\xi, \wp_1, \wp_2, \wp_3)$ is quasi H -closed.

Definition 2.5. [16] If all open cover of $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ has a finite subcollection where its closure covers ξ , then

$\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a p -quasi- H -closed space.

Definition 2.6. When any $\wp_i$ open cover of a tritopological space $(\xi, \wp_1, \wp_2, \wp_3)$ has a finite subcollection with property that its closure's interior covers ξ , then the topological space $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is tri-nearly compact, $\forall i = 1, 2, 3$.

Definition 2.7. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ be a tri-topological space, then if any $\wp_i$ - semi open cover of ξ has a finite subcollection, such that its closure covers ξ , so $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri- S -closed space $\forall i = 1, 2, 3$.

Definition 2.8 A tritopological space $(\xi, \wp_1, \wp_2, \wp_3)$ is called a tri C -compact if for any closed subset A of ξ and if any $\wp_i$ - open cover $W_{\sim} = \{w_{\alpha} : \alpha \in \lambda\}$, there exists a $\wp_j$ - open sets $\{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_n}\}$ such that $A \subset \bigcup_{i=1}^n \overline{w_{\alpha_i}}$, for all $i, j = 1, 2, 3, i \neq j$.

Definition 2.9 A tri- Hausdorff space $(\xi, \wp_1, \wp_2, \wp_3)$ is called tri - H - closed if $\forall \wp_i$ - open cover $W_{\sim} = \{w_{\alpha} : \alpha \in \lambda\}$ has a finite $\wp_j$ - collection $\{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_n}\}$ such that $w_{\alpha_n} \subset \bigcup_{i=1}^n \overline{w_{\alpha_i}}$, for all $i, j = 1, 2, 3, i \neq j$.

Definition 2.10 A subset A of a tri-topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is called a tri-regular open if $\text{Int}(\bar{A}) = A$ in each topological space $\wp_i, i = 1, 2, 3$.

Theorem 2.11. A tri-topological space $(\xi, \wp_1, \wp_2, \wp_3)$ is called C -compact if and only if for any subset A of ξ and for any $\wp_i$ open cover $W_{\sim} = \{w_{\alpha} : \alpha \in \lambda\}$ of A , where w_{α} is tri-regular open for all $\alpha \in \lambda$, $\exists$ a $\wp_i$ -collection $\{w_{\alpha_k}\}_{k=1}^n$ of $W_{\sim}$ such that $w_{\alpha_k} \subset \bigcup_{i=1}^n \overline{w_{\alpha_i}} \forall i = 1, 2, 3$.

Proof. $\Rightarrow$ Suppose that ξ is a tri- C -compact space and $A \subset \xi$. Let $W_{\sim} = \{w_{\alpha} : \alpha \in \lambda\}$ be a $\wp_i$ open cover of A , such that w_{α} is tri-regular open $\forall i = 1, 2, 3, \forall \alpha \in \lambda$. So we have $[\text{Int}(\overline{w_{\alpha_i}}) = w_{\alpha_i}] \forall i = 1, 2, 3$. Because $\text{Int}(\overline{w_{\alpha_i}})$ is $\wp_i$ - open set $\forall \alpha \in \lambda$, by the tri- C -compactness of ξ , the result valid.

$\Leftarrow$ If $W_{\sim} = \{w_{\alpha} : \alpha \in \lambda\}$ is a $\wp_i$ -open cover of $A, \forall A \subset \xi, \forall i = 1, 2, 3$. So

$$A \subset \bigcup_{\alpha \in \lambda} w_{\alpha} \subset \bigcup_{\alpha \in \lambda} \overline{w_{\alpha}} \subset \bigcup_{\alpha \in \lambda} \text{Int}(\overline{w_{\alpha}}).$$

Since $\{\text{Int}(\overline{w_{\alpha}}), \alpha \in \lambda\}$ constitutes an open cover of A , then let $\text{Int}(\overline{w_{\alpha}}) = r_{\alpha} \forall \alpha \in \lambda$. Therefore, we obtain $A \subset \bigcup_{\alpha \in \lambda} r_{\alpha}$, thus by the conditions, $A \subset \bigcup_{\alpha \in \lambda} r_{\alpha_k}$ and hence ξ is a tri- C -compact space.

2.2 Relation between C - compactness and C -separation axioms in a tritopological space

We continue to derive numerous findings theoretically in this section, but this time we show how the C -compactness in a tritopological space and the C -separation axioms relate to each other.

Definition 2.12. Consider $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri-topological space, then $\forall i, j = 1, 2, 3, i \neq j, \xi$ is:

- i. A $tri-C-T_4$ -space if $\forall g \neq h \in \xi, \exists w_g$ of a $\wp_i$ open in $\xi: g \in \overline{w_g}$ and $h \notin \overline{w_g}$ or $\exists v_h$ of $\wp_j$ -open in which $h \in \overline{v_h}$ and $g \notin \overline{v_h}$.
- ii. A $tri-C-T_1$ -space if $\forall g \neq h \in \xi, \exists w_g$ of a $\wp_i$ -open: $g \in \overline{w_g}, h \notin \overline{w_g}$ and $\exists v_h$ of $\wp_j$ -open in which $g \in \overline{w_g}$ and $h \notin \overline{w_g}$ and $h \in \overline{v_h}$ and $g \notin \overline{v_h}$.
- iii. A $tri-C-T_2$ -space if $\forall g \neq h \in \xi, \exists w_g$ of $\wp_i$ -open set and v_h a $\wp_j$ -open set: $g \in \overline{w_g}$ and $h \in \overline{v_h}$ and $\overline{w_g} \cap \overline{v_h} = \emptyset$.
- iv. A $tri-C$ -regular space if $\forall g \notin A$, where A is a $\wp_i$ closed subset of $\xi, \exists w_g$ of $\wp_i$ -open and v_A of $\wp_j$ -open: $g \in w_g, A \subset v_A$ and $\overline{w_g} \cap \overline{v_A} = \emptyset$.
- v. A $tri-C-T_3$ -space if ξ is a $tri-C-T_1$ -space and $tri-C$ - regular space.
- vi. A $tri-C$ -normal space if $\forall A, B \subseteq \wp_i$ -closed sets such that $A \cap B = \emptyset, \exists \wp_i$ - open set and $\wp_j$ -open set: $A \in \overline{w_A}, B \in \overline{v_B}$.
- vii. A $tri-C-T_4$ -space if ξ is $tri-C$ -normal space and $tri-C-T_1$ -space.

Theorem 2.13. Let M be a C -compact subset in $C-T_2$ -space that contains h , then $\forall g \neq h, \exists w_g, v_M$ open sets such that $g \in \overline{w_g}, M \subset \overline{v_M}$ and $\overline{w_g} \cap \overline{v_M} = \emptyset$.

Proof. Consider $m \in M$ and $g \neq a, g \notin M$ and ξ is $tri-C-T_2$ -space, then $\exists w_g(m), v(m)$ two $\wp_i$ -open sets in which $w_g(m)$ and $m \in \overline{v(m)}$ with $\overline{w_g(m)} \cap \overline{v(m)} = \emptyset$ for all $i \neq j, i, j = 1, 2, 3$. Therefore $V_\sim = \{v(m): a \in M\}$ is an open cover of M and $\exists \{v_{\alpha_k}\}_{k=1}^n$ in which $\overline{M} \subset \bigcup_{k=1}^n \overline{v_{\alpha_k}(m)}$, thus we obtain $g \in \overline{v_{\alpha}(m)}$ and $M \subset \overline{v}$ where $\overline{v} = \bigcup_{k=1}^n \overline{v_{\alpha_k}(m)}$, and this implies

$$\overline{w(m)} \cap \overline{v} = \overline{w(m)} \cap \bigcup_{k=1}^n \overline{v_{\alpha_k}(m)} = \bigcup_{k=1}^n \overline{w(m)} \cap \overline{v_{\alpha_k}(m)} = \bigcup_{k=1}^n \emptyset = \emptyset.$$

Theorem 2.14. If N, M are two disjoint $\wp_i$ - $tri-C$ -compact subsets of a $tri-T_2$ -space $\xi = (\xi, \wp_1, \wp_2, \wp_3)$, then they can be separated by two disjoint $\wp_j$ -open sets w_N and v_M in which $N \subset w_N$ and $M \subset v_M \forall i \neq j, i, j = 1, 2, 3$.

Proof. Consider the two disjoint $\wp_i$ - $tri-C$ -compact subset N and M . Consider $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is $tri-T_2$ -space $\forall a \in N$. We have $a \notin M$, since M is a $\wp_i$ - $tri-C$ -compact in ξ . Now, by a previous theorem $\exists B_j$ -open $w(a)$ and $v(M)$ which $a \in w(a)$ and $M \subset v(M)$.

Thus $W_\sim = \{w(a): a \in N\}$ replace an $\wp_i$ -open cover of N , and so $N = \bigcup_{a \in N} w(a)$. Due two N is a $\wp_i$ - $tri-C$ -compact subset, $\exists \{w_k(a): k = 1, 2, \dots, n, w_k(a)\}$ a tri -regular open set in which $N \subset \bigcup_{k=1}^n \overline{Int(w_k(a))}$, (say $\bigcup_{k=1}^n \overline{Int(w_k(a))} = w_N$) for all $i = 1, 2, 3$.

Accordingly, $N \subset w_N$ and $M \subset v_M$ such that w_N and v_M are two $\wp_i$ -open sets.

It is enough to show $w_N \cap v_M = \emptyset$. $w_N \cap v_M = \left(\bigcup_{k=1}^n \overline{Int(w_k(a))} \right) \cap v_M = \bigcup_{k=1}^n \overline{Int(w_k(a))} \cap v_M$,

but $w_k(a) \cap v_M = \emptyset$ and $\overline{Int(w_k(a))} \subset \overline{w_k(a)}$, consequently we have $\overline{Int(w_k(a))} \cap v_M = \emptyset$ and hence $\bigcup_{k=1}^n \overline{Int(w_k(a))} \cap v_M = \bigcup_{k=1}^n \emptyset = \emptyset$.

Definition 2.15. If every open set is $\wp_i$ -clopen, then $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is tri -extremely disconnected $\forall i = 1, 2, 3$.

Theorem 2.16. The space $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is tri -extremely disconnected compact if and only if it is $ri-C$ compact.

Proof: If $\{w_\alpha: \alpha \in \lambda\}$ is an open cover of $\xi, \forall i = 1, 2, 3, \xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri -extremely disconnected compact space, and $N \subseteq \xi$, then $N = \bigcup_{\alpha \in \lambda} w_\alpha$ and $\xi_e = (\xi_e - N) \cup N$.

Hence, $\xi_e = (\xi_e - N) \cup \left(\bigcup_{\alpha \in \lambda} w_\alpha \right)$

As a result, $\xi_e = \{\xi_e - N; w_\alpha: \alpha \in \lambda\}$ covers ξ due to ξ is a tri -compact space.

So, ξ has a $\wp_i$ -finite subcover (say ξ_e) where $\xi_e = (\xi_e - N) \cup \left(\bigcup_{\alpha \in \lambda} w_{\alpha_k} \right), k = 1, 2, 3$.

Consequent, $\xi_e = (\xi_e - N) \cup \left(\bigcup_{\alpha \in \lambda, k=1} w_{\alpha_k} \right)$

Thus, $N = \bigcup_{\alpha \in \lambda, k=1} w_{\alpha_k}$ due to ξ is a tri -disconnected space then $\overline{w_{\alpha_k}} = w_{\alpha_k}$.

So, $N = \bigcup_{k=1}^n \overline{w_{\alpha_k}}$ for ξ is a $tri-C$ -compact space. Now, consider $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri -extremely disconnected C -compact space. If $W_\sim = \{w_\alpha: \alpha \in \lambda\}$ is a $\wp_i$ -open cover of ξ and $N \subset \xi$, so W is a $\wp_i$ -open cover of N . Consequently, $N = \bigcup_{k=1}^n w_{\alpha_k}$. But we have $\xi = N \cup (\xi - N) = \left(\bigcup_{k=1}^n w_{\alpha_k} \right) \cup (\xi - N)$,

so $\{\xi_e - N; w_{\alpha_k}: k = 1, 2, \dots, n\}$ is a finite subcover of ξ . Hence the result.

Theorem 2.17. Each tri -compact space is a $tri-C$ -compact space.

Proof. Let $W_\sim = \{w_\alpha: \alpha \in \lambda\}$ be a $\wp_i$ -open cover of N is a subset of ξ . So $\{w_\alpha: \xi - N: \alpha \in \lambda\}$ constitutes a $\wp_i$ -open cover of $\xi \forall i = 1, 2, 3$. With respect to $\xi, \xi \subset \bigcup_{k=1}^n w_{\alpha_k} \subset (\xi - N)$ and

so $N \subset \bigcup_{k=1}^n w_{\alpha_k} \subset \bigcup_{k=1}^n \overline{w_{\alpha_k}}$. Therefore $\{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_n}\}$ is a collection of w and $\subset \bigcup_{k=1}^n \overline{w_{\alpha_k}}$. Hence, ξ is a $tri - C - compact$ space.

Theorem 2.18. Each *triextremely disconnected nearly compact* space is a $tri - C - compact$ space.

Proof. Consider $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is $tri - extremely disconnected nearly compact$, $N \subset \xi$, and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is $\wp_i - open$ cover of N . So, $N \subset \bigcup_{k=1}^n w_{\alpha_k}$ and $\xi - N$ covers it set. So, we have $\{w_\alpha, \xi - N : \alpha \in \lambda\}$ constitutes an open cover of $\forall i = 1, 2, 3$. Now, ξ is $tri - extremely disconnected$, that implies w_α is a $\wp_i - clopen$ set $\forall \alpha \in \lambda$, therefore $w_\alpha = \overline{w_\alpha}$ and so $w_\alpha^\circ = \overline{w_\alpha}$.

Thus, we get $w_\alpha^\circ = w_\alpha$ and consequently, we obtain $\xi \subset (\bigcup_{\alpha \in \lambda} \overline{w_\alpha^\circ}) \cup (\xi - N)$.

Now, since it is a $tri - nearly compact$ space that $\xi \subset (\bigcup_{\alpha \in \lambda} \overline{w_\alpha^\circ}) \cup (\xi - N)$ this immediately given $N \subset \bigcup_{\alpha \in \lambda} \overline{w_\alpha^\circ}$. Hence ξ is $tri - C - compact$.

Theorem 2.19. Each quasi $tri - H - closed$ space is a $tri - C - compact$ space.

Proof. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a $tri - quasi - H - closed$ space, $N \subset \xi$ and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is a $\wp_i$ open cover of $\forall i = 1, 2, 3$, then $\{w_\alpha, \xi - N : \alpha \in \lambda\}$ composes a tri -open cover.

$\xi \subset (\bigcup_{\alpha \in \lambda, k=1}^n \overline{w_{\alpha_k}}) \cup (\xi - N)$. also, since $(\xi - N)$ cover $\xi - N$ then $N \subset \bigcup_{\alpha \in \lambda, k=1}^n \overline{w_{\alpha_k}}$ and hence ξ is a tri - C -compact space.

Theorem 2.20. If $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a $tri - H - closed$ and $tri - S - closed$ space, then ξ is $C - compact$ space.

Proof. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ be a tri - H -closed and tri - S -closed space. If $N \subseteq \xi$ and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is $\wp_i$ -open cover of $\forall i = 1, 2, 3$. Then $\exists \alpha \in \lambda$ in which $w_\alpha \subset N$ as ξ is a tri - c -closed space. As a consequence, $\bigcup_{\alpha \in \lambda} \overline{w_\alpha} \subset N \subset \bigcup_{\alpha \in \lambda} \overline{w_\alpha}$. Therefore $\{\overline{w_\alpha} : \alpha \in \lambda\}$ forms a $\wp_i$ - H -closed cover of ξ because of the tri - H -closed space, then $N \subset \bigcup_{\alpha \in \lambda} \overline{w_\alpha}$. Hence it is a tri - C -compact space.

Theorem 2.21. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ be a *triextremely disconnected* space, then the following are equivalent:

- i. ξ is $tri - c - compact$ with respect to $\wp_i$ closed subspace.
- ii. ξ is nearly compact.
- iii. ξ is $tri - quasi - H - space$.

Proof. i $\rightarrow$ ii: Let ξ be tri - c -compact and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ be a $\wp_i$ -open cover of $\xi \forall i = 1, 2, 3$. Now, $\forall \wp_i$ closed set $B \subset \xi$, $W_\sim$ is a $\wp_i$ cover of B , so $B \subset \bigcup_{k=1}^n w_{\alpha_k}$.

But it is $tri - c$ -compact, then $\subset \bigcup_{k=1}^n w_{\alpha_k}$. Since ξ is a tri -extremely disconnected space, $\xi - B$ and w_α are $\wp_i$ -clopen set. Thus, one might get $\overline{w_\alpha^\circ} = w_{\alpha_k} \forall k = 1, 2, \dots, n$. As a result $\subset \overline{w_{\alpha_k}^\circ}$, this immediately yields $\xi \subset (\bigcup_{k=1}^n \overline{w_{\alpha_k}^\circ} \cup (\xi - B))$. So, $(\xi - B)$, is a $\wp_i$ -subcover of $W_\sim$ interior of closure of $\wp_i$ -open that covers ξ . Thus, ξ is a $tri - nearly compact$ space.

ii $\rightarrow$ iii: If ξ is a $tri - nearly compact$ space and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is a $\wp_i$ -open cover of ξ , then if processes are finite $\wp_i$ -subcover of the interior of the closure set, say $\overline{w_{\alpha_k}^\circ}, (\xi - B)$, $k = 1, 2, \dots, n$ by $tri - nearly compactness$, $\xi \subset (\bigcup_{k=1}^n \overline{w_{\alpha_k}^\circ} \cup (\xi - B))$ since ξ is a $tri - extremely disconnected$ space, then w_{α_k} is a $\wp_i$ -clopen set. Hence $\overline{w_{\alpha_k}^\circ} = \overline{w_{\alpha_k}}$ which consequently leads to $\xi \subset (\bigcup_{k=1}^n \overline{w_{\alpha_k}})$. Therefore, ξ is a tri - H -close space.

iii $\rightarrow$ i: If ξ is $tri - H - closed$, and $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is a $\wp_i$ -open cover of $B : B$ is $\wp_i$ -closed subspace of ξ , therefore $W_\sim$ is a $\wp_i$ -open cover of B due to ξ is a $tri - H - closed$ space, then $B \subset \xi \subset \overline{w_\alpha}$.

Also, because ξ is a $tri - extremely disconnected$ space, then $B \subset \overline{w_\alpha}$ and hence ξ is a $tri - C - compact$ space.

Theorem 2.22. The $tri - C - compactness$ has a hereditary property with respect to $\wp_i - closed$ subspace $\forall i = 1, 2, 3$.

Proof. If ξ is a tri - C -compact space, if B is a closed subspace of ξ and $D \subseteq B$. Consider $W_\sim = \{w_\alpha : \alpha \in \lambda\}$ is an $\wp_i$ -open cover of B , then $\xi = D \cup (\xi - D)$. Since $D \subset B$, then $\xi = D \cup (\xi - B)$. Consequently, $\xi = \bigcup_{\alpha \in \lambda} w_\alpha \cup (\xi - D)$, hence $\{w_\alpha, \xi - B : \alpha \in \lambda\}$ forms open cover. Now since a $\wp$ - C -compact space, then every subset can be covered by a finite $\wp_i$ -subcover of closure of subset of $W_\sim$.

So, $B \subset \bigcup_{k=1}^n \overline{w_{\alpha_k}}$. B is at tri - C -compact space.

Theorem 2.23. If $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri - C - T_2 -space tri - c and tri - C -extremely disconnected space. Then ξ is tri - C - T_4 .

Proof. Let $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ be a $tri - c - compact$ and tri - C - T_2 -space. $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a tri - C - T_1 -space. Let C and D be two $\wp_i$ -closed subsets of: $C \cap D = \emptyset, \forall i = 1, 2, 3$, then using 2.22, C and D are $\wp_i -$

C –compact subsets of $tri-C-T_2$. Hence, $\exists$ two $\wp_i - C$ –open sets w_C and v_D in which $C \subset \bar{w}_C$ and $D \subset \bar{v}_D$ with $\bar{w}_C \cup \bar{v}_D = \emptyset$. Then, ξ is $tri - C-T_4$.

Theorem 2.24. If $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a $tri-C-T_2$ and $tri - C$ –extremely disconnected space, then each subset of ξ is a $\wp_i - closed$ set $\forall i = 1, 2, 3$.

Proof. If C is $\wp_i - C$ –compact subset of ξ and $\emptyset \notin C$, then by 2.23, $\exists$ two $\wp_i - C$ –open set $w_\emptyset$ and v_C in which $\emptyset, w_\emptyset$ and $C \in v_C$ with $w_\emptyset \cup v_C = \emptyset$. Therefore, we have $w \subset (v_C)^c$.

Since $\subset v_C$, $v_C^c \subset C^c$. So, $\emptyset \subset w_\emptyset \subset v_C^c \subset C^c$. Also, since $\wp_i -$ is open, C^c is a $\wp_i$ –open set. Thus C is a $\wp_i - closed$ set.

Theorem 2.25. If $\xi = (\xi, \wp_1, \wp_2, \wp_3)$ is a $tri - C$ –compact, $tri-C-T_2$ –space, and $tri - C$ –extremely disconnected space, then every subset of ξ is $tri-C$ –compact if and only if it is a $C - \wp_i$ –closed set $\forall i = 1, 2, 3$.

Proof. ($\rightarrow$) If C is $tri - C$ –compact subset of ξ , then by 2.24, C is $C - \wp_i$ –closed set.

($\leftarrow$) Let C be a $C - \wp_i$ –closed of a $tri - C$ –compact, $C-T_2$ –extremely disconnected space. So, by 2.24, C is a $C - \wp_i$ –compact.

III. CONCLUSION

This study provides a clear definition of the so-called c –compactness in tritopological, bitopological, and topological spaces. Several features of these spaces with their links with other topological. Thus, hypothetically, bitopological and tripartite spaces were formed. Future research is left to determine whether the inferred conclusions can lead to the derivation of other innovative theorems pertaining to the finite product and mappings of tripartite expandable spaces, feebly pairwise expandable spaces, and fuzzy tripartite topological spaces.

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