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Convergence Theorems for a Finite Family of Operators Satisfying MZ-Type Condition In CAT(0) Spaces

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Abstract: In this paper, we prove a convergence result for a finite family of operators satisfying the MZ-type condition in $CAT(0)$ space. This paper aims to study an implicit iterative process for a finite family of operators satisfying the MZ-type condition. Our results improve and extend some corresponding recent results from the current existing literature.

Keywords: $CAT(0)$ space; implicit iteration; Zamfirescu operator; Ciric quasi-contraction; MZ-condition.

Introduction and Preliminaries:

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, $J = \{1,2, \ldots, N\}$, be the initial segment of positive integers, and $F(T)$, be the set of fixed points of T . Let

$$F = \cap_{j=1}^{N} F(T_j),$$

be the set of common fixed points of finite families of mappings $\{T_j : j \in J\}$.

Assume that $(X, d)$is a metric space. A geodesic path between $\theta \in X$ and $\eta \in X$ (or, to put it another way, a geodesic from θ to η) is a map r from $[0, l]$ to X with $r(0) = r(l)$ and $d(r(t), r(t_0)) = |t - t_0|$, for any $t, t_0 \in [0, l]$. Thus r is an isometry and $d(\theta, \eta) = l$. The image of r is a geodesic (or metric) segment that joins θ and η. When geodesic is unique, it is denoted by $[\theta, \eta]$.

A metric space is called a geodesic if any pair of its points can be joined by a geodesic. If D contains every geodesic segment connecting any two points, the subset $D \subseteq X$ is convex.

A geodesic triangle $\triangle(\theta_1, \theta_2, \theta_3)$ in a geodesic metric space (X, d) is made up of three points $\theta_1, \theta_2, \theta_3$ in X (the vertices of $\triangle$, with a geodesic segment connecting each pair of vertices (the edge of $\triangle$).

A comparison triangle for the geodesic triangle $\triangle(\theta_1, \theta_2, \theta_3)$ in $(X, d)$is a triangle $\overline{\triangle}(\theta_1, \theta_2, \theta_3) = \triangle(\overline{\theta_1}, \overline{\theta_2}, \overline{\theta_3})$ in Euclidean space $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{\theta_i}, \overline{\theta_j}) = d(\theta_i, \theta_j)$ for $i, j \in \{1,2,3$ [4].

If the distance between any two points on a geodesic triangle $\triangle$ does not exceed the distance between its corresponding pair of points on its comparison triangle $\overline{\triangle}$, then the geodesic space X is a $CAT(0)$ space.

Let $\overline{\triangle}$ be a comparison triangle for a geodesic triangle $\triangle$ in X. The $\triangle$ is satisfy the CAT(0) inequality if for all $\theta, \eta \in \triangle$ and its comparison points $\overline{\theta}, \overline{\eta} \in \overline{\triangle}$ such that

$$d(\theta, \eta) \leq d_{\mathbb{R}^2}(\overline{\theta}, \overline{\eta}).$$

A complete CAT(0) space is often called Hadamard space [3].

Let θ, η, ζ are points of X and η_0 be the midpoint of the segment $[\eta, \zeta]$, denoted by $\frac{\eta \oplus \zeta}{2}$, then the CAT(0) inequality gives

$$d^2(\theta, \eta_0) \leq \frac{1}{2} d^2(\theta, \eta) + \frac{1}{2} d^2(\theta, \zeta) - \frac{1}{4} d^2(\eta, \zeta).$$

This is called the (CN) inequality of Bruhat and Tits [2]. A geodesic space is said to be $CAT(0)$ space if and only if it satisfies the (CN) inequality [4].

Kirk [6] first studied fixed point theory in CAT(0) spaces. He proved that every nonexpansive(single-valued) mapping defined on a closed, bounded convex subset of a complete CAT(0) space always had a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in $CAT(0)$spaces has been rapidly developed, and many papers have appeared(e.g., [14]-[19]).

It is worth mentioning that the results $CAT(0)$ spaces can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k' \geq k$ [4].

Definition 1.1. Let (X, d) be a $CAT(0)$ space and D be its nonempty subset of X. Then $T: D \to D$ is said to be

1. α contraction, if $d(T\theta, T\eta) \leq \alpha d(\theta, \eta)$, for all $\theta, \eta \in D$, $0 \leq \alpha < 1$;
2. nonexpansive, if $d(T\theta, T\eta) \leq d(\theta, \eta)$, for all $\theta, \eta \in D$;
3. quasi-nonexpansive, $d(T\theta, p) \leq d(\theta, p)$, for all $\theta \in D$, $p \in F(T)$;
4. Kannan mapping [10], if there exists $b \in \left[0, \frac{1}{2}\right)$ such that

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$$d(T\theta, T\eta) \le b[d(\theta, T\theta) + d(\eta, T\eta)], \text{ for all } \theta, \eta \in D;$$

5. Chatterjea mapping [11], if there exists $c \in \left[0, \frac{1}{2}\right)$ such that

$$d(T\theta, T\eta) \le c[d(\theta, T\eta) + d(\eta, T\theta)], \text{ for all } \theta, \eta \in D.$$

Example 1.1. Let $X = \mathbb{R}$, $D = [0,1]$ with the usual metric $d(\theta, \eta) = |\theta - \eta|$.
We define an operator $T: [0,1] \to [0,1]$ such that

$$T(\theta) = \begin{cases} \frac{4}{7}, & \theta \in [0, \frac{3}{4}) \\ \frac{3}{7} & \theta \in [\frac{3}{4}, 1] \end{cases}.$$

Then T is a discontinuous Kannan operator.

Proof: Since Kannan condition(K) is

$$|T\theta - T\eta| \le b\,[\,|\theta - T\theta| + |\eta - T\eta|], \qquad \text{(K)}$$

for any $\theta, \eta \in [0,1]$.
We have to discuss the following cases:

I. $\theta, \eta \in \left[0, \frac{3}{4}\right)$ or $\theta, \eta \in \left[\frac{3}{4}, 1\right]$. Then $T(\theta) = T(\eta)$, the left-hand side of (K) becomes 0 and consequently (K) holds for any $b \in \left[0, \frac{1}{2}\right)$.

II. $\theta \in \left[0, \frac{3}{4}\right)$ and $\eta \in \left[\frac{3}{4}, 1\right]$. Then $T(\theta) = \frac{4}{7}$, $T(\eta) = \frac{3}{7}$ and (K) becomes

$$\frac{1}{7} \le b\left[\left|\theta - \frac{4}{7}\right| + \left|\eta - \frac{3}{7}\right|\right] \qquad (1.1)$$

As $0 \le \theta < \frac{3}{4}$, it follows that $\frac{-4}{7} \le \theta - \frac{4}{7} \le \frac{5}{28}$, so $\left|\theta - \frac{4}{7}\right| \in \left[0, \frac{4}{7}\right]$.
As $\frac{3}{4} \le \eta \le 1$, it follows that $\frac{9}{28} \le \eta - \frac{3}{7} \le \frac{4}{7}$, so $\left|\eta - \frac{3}{7}\right| \in \left[\frac{9}{28}, \frac{4}{7}\right]$.
It follows that the minimum value of the right-hand side in (K) is $b\frac{9}{28}$, so for (K) to hold it is necessary that $\frac{1}{7} \le b\,\frac{9}{28}$, which implies $b \ge \frac{4}{9}$.

III. $\eta \in \left[0, \frac{3}{4}\right)$ and $\theta \in \left[\frac{3}{4}, 1\right]$. Similarity to case II, due to the symmetry of (K), it follows that $b \ge \frac{4}{9}$.

From I, II, and III, we conclude that T satisfy (K) with $b \in \left[\frac{4}{9}, \frac{1}{2}\right)$.

Example 1.2. Let $X = \mathbb{R}$, $D = [0,1]$ with the usual metric $d(\theta, \eta) = |\theta - \eta|$.
We define an operator $T: [0,1] \to [0,1]$ such that

$$T(\theta) = \begin{cases} \frac{1}{4}, & \theta \in [0, \frac{1}{2}) \\ \frac{1}{5} & \theta \in [\frac{1}{2}, 1] \end{cases}.$$

Then T is discontinuous Chatterjea operator.

Proof: Since Chatterjea condition(C) is

$$|T\theta - T\eta| \le c[|\theta - T\eta| + |\eta - T\theta|], \qquad \text{(C)}$$

for any $\theta, \eta \in [0,1]$.
We have to discuss the following cases:

I. $\theta, \eta \in \left[0, \frac{1}{2}\right)$ or $\theta, \eta \in \left[\frac{1}{2}, 1\right]$.

Then $T(\theta) = T(\eta)$, the left-hand side of (C) becomes 0 and consequently (C) holds for any $b \in \left[0, \frac{1}{2}\right)$.

II. $\theta \in \left[0, \frac{1}{2}\right)$ and $\eta \in \left[\frac{1}{2}, 1\right]$.

Then $T(\theta) = \frac{1}{4}$, $T(\eta) = \frac{1}{5}$ and (C) becomes

$$\frac{1}{20} \le c\left[\left|\theta - \frac{1}{5}\right| + \left|\eta - \frac{1}{4}\right|\right]. \qquad (1.2)$$

As $0 \le \theta < \frac{1}{2}$, it follows that $-\frac{1}{5} \le \theta - \frac{1}{5} \le \frac{3}{10}$, so $\left|\theta - \frac{1}{5}\right| \in \left[0, \frac{1}{5}\right]$.
As $\frac{1}{2} \le \eta \le 1$, it follows that $\frac{1}{4} \le \eta - \frac{1}{4} \le \frac{3}{4}$, so $\left|\eta - \frac{1}{4}\right| \in \left[\frac{1}{4}, \frac{3}{4}\right]$.
It follows that the minimum value of the right-hand side in (C) is $\frac{c}{4}$, so for (C) to hold it is necessary that $\frac{1}{20} \le \frac{c}{4}$, which implies $c \ge \frac{1}{5}$.

III. $\eta \in \left[0, \frac{1}{2}\right)$ and $\theta \in \left[\frac{1}{2}, 1\right]$.

Similarity to case II, due to the symmetry of (C), it follows that $c \ge \frac{1}{5}$.

From I, II, and III, we conclude that T satisfy (C) with $c \in \left[\frac{1}{5}, \frac{1}{2}\right)$.

In 2001, Xu and Ori [7] introduced the implicit iteration procedure that follows for common fixed points inside a finite family of nonexpansive mappings as follows as:

Let D be a nonempty closed, convex subset of a normed space X. Let $\{T_i, i \in J\}$ be N nonexpansive self mapping of D. Let $\theta_0 \in D$ be an arbitrary point and $\sigma_n \subset (0,1)$, then the sequence $\{\theta_n\}$ generated in the compact form as follows:

$$\theta_n = \sigma_n \theta_{n-1} + (1 - \sigma_n) T_n \theta_n, \; n \in N, \qquad (1.3)$$

where $T_n = T_n\ (mod\ N)$(the $mod\ N$ takes value in $J = \{1,2,3, \dots, N\}$). They proved a weak convergence theorem using this process.

In 2004, Osilike [9] extended the results of Xu and Ori [7] from nonexpansive mapping to strictly pseudo contractive mappings.
Inspired by the above facts, in 2006 Su and Li [8] introduced a new two-step implicit iteration process in Banach space as follows:
Let D be a nonempty closed, convex subset of a real Banach space X. Let $\{T_i, i \in J\}$ be N strictly pseudocontractive self-maps of D. Let $\theta_0 \in D$ be an arbitrary point and $\{\sigma_n\}, \{\tau_n\} \subset (0,1)$, then the sequence $\{\theta_n\}$ generated as follows:

$$\begin{cases} \theta_n = \sigma_n \theta_{n-1} + (1-\sigma_n) T_n \eta_n, \\ \eta_n = \tau_n \theta_{n-1} + (1-\tau_n) T_n \theta_n, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $T_n = T_n (mod\ N)$.
They proved a convergence theorem for a finite family of strictly pseudocontractive maps using this iteration.
Let D be a nonempty closed, convex subset of a $CAT(0)$ space X. Let $\{T_i, i \in J\}$ be N self- maps of D. Then iteration (1.4) can be generated for a subset D of $CAT(0)$ space X as follows:

$$\begin{cases} \theta_0 \in D, \\ \theta_n = \sigma_{n\theta_{n-1}} \oplus (1-\sigma_n) T_n \eta_n, \\ \eta_n = \tau_{n\theta_{n-1}} \oplus (1-\tau_n) T_{n\theta_n}, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $T_n = T_n (mod\ N)$ and $\{\sigma_n\}, \{\tau_n\} \subset (a,b)$, for some $a, b \in (0,1), a < b$.
In 1972, Zamfirescu [12] proved the following important Theorem.

Theorem Z. Let (X,d) be a complete metric space and $T: X \to X$ be mapping for which there exist real numbers a, b, and c satisfying $0 < a < 1$, $0 < b, c < \frac{1}{2}$ such that for any $\theta, \eta \in X$, at least one of the following conditions holds:
(Z_1) $d(T\theta, T\eta) \le \alpha d(\theta, \eta)$,
(Z_2) $d(T\theta, T\eta) \le b[d(\theta, T\theta) + d(\eta, T\eta)]$,
(Z_3) $d(T\theta, T\eta) \le c[d(\theta, T\eta) + d(\eta, T\theta)]$.
Then T has a unique fixed point θ and the Picard iteration $\{\theta_n\}_{n=0}^{\infty}$ defined by

$$\theta_{n+1} = T\theta_n,\ n = 0,1,2, \ldots,$$

converges to θ for any arbitrary but fixed $\theta_0 \in$ X.
The conditions $(Z_1) - (Z_3)$ (can be written in the following equivalent form

$$d(T\theta, T\eta) \le h \max\{d(\theta, \eta), \frac{d(T\theta, \theta) + d(\eta, T\eta)}{2}, \frac{d(\theta, T\eta) + d(\eta, T\theta)}{2}\}, \qquad (QC)$$

for all $\theta, \eta \in X$ and $0 < h < 1$, has been obtained by Ciric [13] in 1974.
A mapping satisfying (QC) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (QC).

Definition : (MZ- condition) Let (X,d) be a $CAT(0)$ space and D be the subset of X. A mapping $T: D \to D$ is said to satisfy the MZ condition if there exists $0 < h < 1$, such that

$$d(T\theta, T\eta) \le \alpha d(\theta, \eta) + (1-\alpha) \max\{d(\theta, \eta), \frac{d(T\theta, \theta) + d(\eta, T\eta)}{2}, \frac{d(\theta, T\eta) + d(\eta, T\theta)}{2}\},$$

for all $\theta, \eta \in D$.

Lemma 1.1. [1] Let X be a $CAT(0)$ space.
(a) Let $\theta, \eta \in X$, for each $t \in [0,1]$, there exists a unique point $\zeta \in [\theta, \eta]$ such that [5]

$$d(\theta, \zeta) = t d(\theta, \eta),\ d(\eta, \zeta) = (1-t) d(\theta, \eta); \tag{1.6}$$

We use the notation $(1-t)\theta \oplus t\eta$ for unique point ζ satisfying (2.1).
(b) For all $t \in [0,1]$, and $\theta, \eta, \zeta \in X$ such that [5]

$$d\left(\left((1-t)\theta \oplus\right) t\eta, \zeta\right) \le (1-t) d(\theta, \zeta) + t d(\eta, \zeta); \tag{1.7}$$

(c) For all $t \in [0,1]$, and $\theta, \eta, \zeta \in X$ such that [5]

$$d^2\left(\left((1-t)\theta \oplus\right) t\eta, \zeta\right) \le (1-t) d^2(\theta, \zeta) + t d^2(\eta, \zeta) - t(1-t) d^2(\theta, \eta). \tag{1.8}$$

Main results:
In this section, we establish strong convergence results of an implicit iterative process (1.5) to approximate a common fixed point for a finite family of mappings satisfying the MZ-condition in the CAT(0) spaces. For this, first, we prove a Lemma.

Lemma 2.1. Let D be a nonempty closed and convex subset of a complete $CAT(0)$ space X . Let $\{T_i: D \to D\} (j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (Fdenotes the set of common fixed points of $\{T_1, T_2, 3, \ldots, T_N\}$) satisfying the condition

$$(T_i\theta, T_i\eta) \le \alpha d(\theta, \eta) + (1-\alpha) max\,\{d(\theta, \eta), \frac{d(T_i\theta, \theta) + d(\eta, T_i\eta)}{2}, \frac{d(\theta, T_i\eta) + d(\eta, T_i\theta)}{2}\}, \tag{2.1}$$

$\forall\, n = 1,2,3, \ldots, N$; $\forall\, \theta, \eta \in D$ and $0 < \alpha < 1$. Let sequences $0 < a \le \sigma_n, \tau_n \le b < 1, \forall\, n \in \mathbb{N}$ and sequence $\{\theta_n\}$ defined by the iteration scheme (1.5) then $\lim_{n \to \infty} d(\theta_n, p)$ exists for any $p \in F$.

Proof: Let $p \in F = \cap_{i=1}^{N} F(T_i)$. Since $\{T_i : D \to D\}(j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ satisfying the condition (2.1), then if

$$\begin{aligned} d(T_i\theta, T_i\eta) &\leq \alpha d(\theta,\eta) + (1-\alpha)\left(\frac{d(T_i\theta,\theta) + d(\eta, T_i\eta)}{2}\right) \\ &= \alpha d(\theta,\eta) + \tfrac{(1-\alpha)}{2}\big(d(T_i\theta,\theta) + d(\eta,T_i\eta)\big) \\ &\leq \alpha d(\theta,\eta) + \tfrac{(1-\alpha)}{2}\big(d(T_i\theta,\theta) + d(\eta,\theta) + d(\theta,T_i\theta) + d(T_i\theta,T_i\eta)\big), \end{aligned}$$

this implies

$$\left(1 - \frac{(1-\alpha)}{2}\right) d(T_i\theta,T_i\eta) \leq \alpha d(\theta,\eta) + \frac{(1-\alpha)}{2}\big(2d(T_i\theta,\theta) + d(\eta,\theta)\big)$$

$$\tfrac{(1+\alpha)}{2} d(T_i\theta,T_i\eta) \leq \alpha d(\theta,\eta) + \tfrac{(1-\alpha)}{2}\big(2d(T_i\theta,\theta) + d(\eta,\theta)\big)$$

$$\tfrac{(1+\alpha)}{2} d(T_i\theta,T_i\eta) \leq \tfrac{(1+\alpha)}{2} d(\theta,\eta) + (1-\alpha) d(T_i\theta,\theta)$$

this gives that

$$d(T_i\theta,T_i\eta) \leq d(\theta,\eta) + 2\tfrac{(1-\alpha)}{(1+\alpha)} d(T_i\theta,\theta), \tag{2.2}$$

$\forall\, \theta,\eta \in D$ and for every $i \in J = \{1,2,3,\dots,N\}$.

Similarly, if

$$\begin{aligned} d(T_i\theta, T_i\eta) &\leq \alpha d(\theta,\eta) + (1-\alpha)\left(\frac{d(\theta,T_i\eta) + d(\eta, T_i\theta)}{2}\right) \\ &\leq \alpha d(\theta,\eta) + \tfrac{(1-\alpha)}{2}\big(d(\theta,T_i\theta) + d(T_i\theta,T_i\eta) + d(\eta,\theta) + d(\theta,T_i\theta)\big) \\ &= \alpha d(\theta,\eta) + \frac{(1-\alpha)}{2}\big(2d(\theta,T_i\theta) + d(T_i\theta,T_i\eta) + d(\eta,\theta)\big) \\ &= \frac{(\alpha-1)}{2} d(\theta,\eta) + (1-\alpha) d(\theta,T_i\theta) + \frac{(1-\alpha)}{2} d(T_i\theta,T_i\eta), \end{aligned}$$

this implies that

$$\left(1 - \frac{(1-\alpha)}{2}\right) d(T_i\theta,T_i\eta) \leq \frac{(\alpha-1)}{2} d(\theta,\eta) + (1-\alpha) d(\theta,T_i\theta)$$

$$\tfrac{(1+\alpha)}{2} d(T_i\theta,T_i\eta) \leq \tfrac{(\alpha-1)}{2} d(\theta,\eta) + (1-\alpha) d(\theta,T_i\theta),$$

this gives

$$d(T_i\theta,T_i\eta) \leq d(\theta,\eta) + 2\tfrac{(1-\alpha)}{(1+\alpha)} d(T_i\theta,\theta), \tag{2.3}$$

$\forall\, \theta,\eta \in D$ and for every $i \in J = \{1,2,3,\dots,N\}$.

Similarly, if

$$\begin{aligned} d(T_i\theta,T_i\eta) &\leq \alpha d(\theta,\eta) + (1-\alpha) d(\theta,\eta) \\ &= d(\theta,\eta), \end{aligned}$$

this gives

$$d(T_i\theta,T_i\eta) \leq d(\theta,\eta), \tag{2.4}$$

$\forall\, \theta,\eta \in D$ and for every $i \in J = \{1,2,3,\dots,N\}$.

Now, since $T_ip = p$, $T_n = T_n(mod\ N)$ and the $mod\ N$ takes the values in J, for $\eta = \theta_n$ and $\theta = p$, the inequalities (2.2), (2.3), and (2.4) gives that

$$d(T_i\theta_n, p) \leq d(\theta_n, p). \tag{2.5}$$

Again, with $\eta = \eta_n$ and $\theta = p$ in (2.2), (2.3) and (2.4), we get

$$d(T_i\eta_n, p) \leq d(\eta_n, p). \tag{2.6}$$

Now, let $\{\theta_n\}$ be the implicit iteration process defined by (1.5) and $\theta_0 \in D$ be an arbitrary.

Then

$$\begin{aligned} d(\theta_n,p) &\leq d(\sigma_n\theta_{n-1} \oplus (1-\sigma_n)T_n\eta_n, p) \\ &\leq \sigma_n d(\theta_{n-1},p) + (1-\sigma_n) d(T_n\eta_n,p) \\ &\leq \sigma_n d(\theta_{n-1},p) + (1-\sigma_n) d(\eta_n,p) \\ &= \sigma_n d(\theta_{n-1},p) + (1-\sigma_n) d(\tau_n\theta_{n-1} \oplus (1-\tau_n)T_n\theta_n, p) \\ &\leq \sigma_n d(\theta_{n-1},p) + (1-\sigma_n)\tau_n d(\theta_{n-1},p) + (1-\sigma_n)(1-\tau_n) d(T_n\theta_n,p) \\ &\leq [\sigma_n + (1-\sigma_n)\tau_n] d(\theta_{n-1},p) + (1-\sigma_n)(1-\tau_n) d(\theta_n,p), \end{aligned}$$

this implies

$$\begin{aligned} d(\theta_n,p) &\leq \frac{(\sigma_n + (1-\sigma_n)\tau_n)}{\big(1-(1-\sigma_n)(1-\tau_n)\big)} d(\theta_{n-1},p) \\ &= \tfrac{(\sigma_n + (1-\sigma_n)\tau_n)}{(\sigma_n + (1-\sigma_n)\tau_n)} d(\theta_{n-1},p) \end{aligned}$$

this gives

$$d(\theta_n,p) \leq d(\theta_{n-1},p), \quad n \in \mathbb{N}. \tag{2.7}$$

This implies that the sequence $\{d(\theta_n,p)\}$ is non-increasing and bounded, hence it is convergent.

Theorem 2.2. Let D be a nonempty closed and convex subset of a complete $CAT(0)$ space X . Let $\{T_i: D \to D\}(j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (Fdenotes the set of common fixed points of $\{T_1, T_2, 3, \ldots, T_N\}$) satisfying the condition

$$(T_i\theta, T_i\eta) \leq \alpha d(\theta,\eta) + (1-\alpha)max\,\{d(\theta,\eta), \frac{d(T_i\theta,\theta)+d(\eta,T_i\eta)}{2}, \frac{d(\theta,T_i\eta)+d(\eta,T_i\theta)}{2}\}$$

$\forall\, n = 1,2,3,\ldots,N;\ \forall\, \theta,\eta \in D$ and $0 < \alpha < 1$. Let sequences $0 < a \leq \sigma_n, \tau_n \leq b < 1, \forall\, n \in \mathbb{N}$ and sequence $\{\theta_n\}$ defined by the iteration scheme (1.5), then $\lim_{n\to\infty} d(T\theta_n, \theta_n)$ exists.

Proof: Let $p \in F,$ then we have already proved that $\lim_{n\to\infty} d(\theta_n, p)$ exists in Lemma 2.1. Assume that

$$\lim_{n\to\infty} d(\theta_n, p) = h, h \geq 0. \qquad (2.8)$$

Now, using (1.5), (2.5), and Lemma 1.2(c), we have

$$\begin{aligned}
d^2(\eta_n, p) &= d^2(\tau_n\theta_{n-1} \oplus (1-\tau_n)T_n\theta_n, p)\\
&\leq \tau_n d^2(\theta_{n-1}, p) + (1-\tau_n)d^2(T_n\theta_n, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n)\\
&\leq \tau_n d^2(\theta_{n-1}, p) + (1-\tau_n)d^2(\theta_n, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n)\\
&\leq \tau_n d^2(\theta_{n-1}, p) + (1-\tau_n)d^2(\theta_{n-1}, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n)\\
&= d^2(\theta_{n-1}, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n),
\end{aligned}$$

this gives

$$d^2(\eta_n, p) \leq d^2(\theta_{n-1}, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n). \qquad (2.9)$$

Using (1.5), (2.6), (2.9), and Lemma 1.2 (c), we have

$$\begin{aligned}
d^2(\theta_n, p) &= d^2(\sigma_n\theta_{n-1} \oplus (1-\sigma_n)T_n\eta_n, p)\\
&\leq \sigma_n d^2(\theta_{n-1}, p) + (1-\sigma_n)d^2(T_n\eta_n, p) - \sigma_n(1-\sigma_n)d^2(\theta_{n-1}, T_n\eta_n)\\
&\leq \sigma_n d^2(\theta_{n-1}, p) + (1-\sigma_n)d^2(\eta_n, p) - \tau_n(1-\sigma_n)d^2(\theta_{n-1}, T_n\eta_n)\\
&\leq \sigma_n d^2(\theta_{n-1}, p) + (1-\sigma_n)\big(d^2(\theta_{n-1}, p) - \tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n)\big)\\
&= \sigma_n d^2(\theta_{n-1}, p) + (1-\sigma_n)d^2(\theta_{n-1}, p) - (1-\sigma_n)\tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n)\\
&= d^2(\theta_{n-1}, p) - (1-\sigma_n)\tau_n(1-\tau_n)d^2(\theta_{n-1}, T_n\theta_n),
\end{aligned}$$

this implies

$$d^2(\theta_{n-1}, T_n\theta_n) \leq \frac{1}{(1-\sigma_n)\tau_n(1-\tau_n)}[d^2(\theta_{n-1}, p) - d^2(\theta_n, p)],$$

this gives

$$d^2(\theta_{n-1}, T_n\theta_n) \leq \frac{1}{a(1-b)^2}[d^2(\theta_{n-1}, p) - d^2(\theta_n, p)] \qquad (2.10)$$

Since $\lim_{n\to\infty} \mathrm{d}(\theta, \mathrm{p}) = \mathrm{h}$, then equation (2.10), gives that

$$\lim_{n\to\infty} d^2\,(\theta_{n-1}, T_n\theta_n) = 0,$$

this implies

$$\lim_{n\to\infty} d\,(\theta_{n-1}, T_n\theta_n) = 0,$$

Therefore

$$\lim_{n\to\infty} d\,(\theta_n, T_n\theta_n) = 0. \qquad (2.11)$$

Definition 2.1. Let D be a nonempty subset of a $CAT(0)$ space (X, d). Then a mapping $T: D \to D$ satisfy condition (I) , if there exists a non-decreasing function $\xi: [0,\infty) \to [0,\infty)$ such that $\xi(0) = 0, \xi(t) \geq 0,\ \forall\, t \in (0,\infty)$ and $d(\theta, T\theta) \geq \xi\big(d(\theta, F(T))\big),\ \forall\, \theta \in D$, where $d(\theta, F(T)) = \inf_{p\in F(T)} d(\theta, p)$, $p \in F(T)$ = set of fixed points of T .

Theorem 2.3. Let D be a nonempty closed and convex subset of a complete $CAT(0)$ space X . Let $\{T_i: D \to D\}(j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (Fdenotes the set of common fixed points of $\{T_1, T_2, 3, \ldots, T_N\}$) satisfying the condition

$$(T_i\theta, T_i\eta) \leq \alpha d(\theta,\eta) + (1-\alpha)max\,\{d(\theta,\eta), \frac{d(T_i\theta,\theta)+d(\eta,T_i\eta)}{2}, \frac{d(\theta,T_i\eta)+d(\eta,T_i\theta)}{2}\}$$

$\forall\, n = 1,2,3,\ldots,N;\ \forall\, \theta,\eta \in D$ and $0 < \alpha < 1$. Let sequences $0 < a \leq \sigma_n, \tau_n \leq b < 1, \forall\, n \in \mathbb{N}$ and sequence $\{\theta_n\}$ defined by the iteration scheme (1.5). If $T_i, i = 1,2,\ldots,N$ are satisfying condition (I), then the sequence $\{\theta_n\}$ is strongly converges to a point $p \in F$.

Proof: From Lemma 2.1, we have $\lim_{n\to\infty} d(\theta_n, p)$ exists for all $p \in F.$

Suppose $\lim_{n\to\infty} d(\theta_n, p) = h,$ for some $h \geq 0.$

Since

$$d(\theta_{n+1}, p) \leq d(\theta_n, p), \forall\ n \in F,$$

taking $\inf_{p\in F}$ in both sides of the above inequality, we have

$$\inf_{p\in F} d(\theta_{n+1}, p) \leq \inf_{p\in F} d(\theta_n, p),$$

yields the inequality

$$d(\theta_{n+1}, F) \leq d(\theta_n, F).$$

This implies that $\{d(\theta_n, F)\}$ is non-decreasing and bounded, hence $\lim_{n\to\infty} d(\theta_n, F)$ exists.

ssSince $\{T_n, n \in J\}$ satisfies the condition (I), then

$$\lim_{n\to\infty} \xi\big(d(\theta_n, F)\big) \le \lim_{n\to\infty} d(\theta_n, T_n\theta_n) \tag{2.12}$$

Also from Theorem 2.2, we have

$$\lim_{n\to\infty} d(\theta_n, T_n\theta_n) = 0.$$

So, inequality (2.12) gives

$$\lim_{n\to\infty} \xi\big(d(\theta_n, F)\big) = 0. \tag{2.13}$$

Since the mapping $\xi: [0,\infty) \to [0,\infty)$ is non-decreasing with $\xi(0) = 0, \xi(t) \ge 0, \forall\, t \in (0,\infty)$.
Now from (2.13), we have

$$\lim_{n\to\infty} d\,(\theta_n, F) = 0.$$

Next, we prove that the sequence $\{\theta_n\}$ is a Cauchy sequence in D. Since $\lim_{n\to\infty} d\,(\theta_n, F) = 0$, so for any given $\epsilon > 0$, there $\exists\; n_\epsilon \in \mathbb{N}$ such that

$$d(\theta_n, F) < \frac{\epsilon}{4},\ \forall\, n \ge n_\epsilon.$$

In particular,

$$\inf\{\, d\big(\theta_{n_\epsilon}, s\big) : s \in F\} < \frac{\epsilon}{4}.$$

So, there must be $q \in F$ such that

$$d\big(\theta_{n_\epsilon}, q\big) < \frac{\epsilon}{2}.$$

Now, for any $n, m \ge n_\epsilon$, we have

$$\begin{aligned} d(\theta_{n+m}, \theta_n) &\le d(\theta_{n+m}, q) + d(\theta_n, q) \\ &\le 2d\big(\theta_{n_\epsilon}, q\big) \\ &< 2\tfrac{\epsilon}{2} = \varepsilon. \end{aligned}$$

Thus sequence $\{\theta_n\}$ is a Cauchy sequence in D. Since D is closed in (X, d) , hence D is complete and hence $\{\theta_n\}$ must converge to the point $p \in D$. Since $\lim_{n\to\infty} d(\theta_n, F) = d(p, F) = 0$ this implies that $p \in F$ that is $\{\theta_n\}$ converges strongly to common fixed points of $T_1, T_2, T_3, \dots, T_N$.

Corollary 2.1. Let D be a nonempty closed and convex subset of a complete $CAT(0)$ space X . Let $\{T_i: D \to D\}(j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (Fdenotes the set of common fixed points of $\{T_1, T_2, 3, \dots, T_N\}$) satisfying the condition

$$d(T_i\theta, T_i\eta) \le \alpha d(\theta, \eta) + (1-\alpha)\left[\frac{d(T_i\theta, \theta) + d(\eta, T_i\eta)}{2}\right],$$

$\forall\, n = 1,2,3,\dots,N;\ \forall\, \theta, \eta \in D$ and $0 < \alpha < 1$.
Let sequences $0 < a \le \sigma_n, \tau_n \le b < 1, \forall\, n \in \mathbb{N}$ and sequence $\{\theta_n\}$ defined by the iteration scheme (1.5). If $T_i, i = 1,2,\dots,N$ are satisfying condition (I), then the sequence $\{\theta_n\}$ is strongly converges to a point $p \in F$.

Corollary 2.2. Let D be a nonempty closed and convex subset of a complete $CAT(0)$ space X . Let $\{T_i: D \to D\}(j \in J)$ be N operators with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (Fdenotes the set of common fixed points of $\{T_1, T_2, 3, \dots, T_N\}$) satisfying the condition

$$d(T_i\theta, T_i\eta) \le \alpha d(\theta, \eta) + (1-\alpha)\left[\frac{d(\theta, T_i\eta) + d(\eta, T_i\theta)}{2}\right],$$

$\forall\, n = 1,2,3,\dots,N;\ \forall\, \theta, \eta \in D$ and $0 < \alpha < 1$.
Let sequences $0 < a \le \sigma_n, \tau_n \le b < 1, \forall\, n \in \mathbb{N}$ and sequence $\{\theta_n\}$ defined by the iteration scheme (1.5). If $T_i, i = 1,2,\dots,N$ are satisfying condition (I), then the sequence $\{\theta_n\}$ is strongly converges to a point $p \in F$.

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