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Physical Formulation and Mathematical Analysis of the Production Decline Curve in Calculation of the Decline Rate for Oil Fields

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Abstract: - The study of fluid filtration in porous media, the different stages in which production passes during the exploitation of oil and gas-bearing fields, the mechanisms in which fluid filtration occurs in porous spaces, as well as the pressure (energy) in the reservoir, which also determines the stage in which the underground reservoir of oil and gas has passed, are the main parameters in the application of many calculations in reservoir engineering. In this paper, the decline analysis method is reflected, specifically in the determination of the Production Decline Rate by analyzing each formula that corresponds to each of the types of decline, including Exponential Decline, Harmonic Decline and Hyperbolic Decline. The main objective of this paper is to present a link between the physical context and the mathematical analysis of production decline curves during a certain time period of the exploitation of underground oil and gas reservoirs, helping to determine the production decline rate.

Keywords: Reservoir, Pressure, Decline Rate, Producing Life, Reservoir, Filtration, Porous Media.

I. Introduction

The flow of fluids that occurs in underground oil and gas bearing reservoir is influenced by its shape. Regardless of the irregular shapes of reservoir boundaries, their rigorous mathematical description is possible using of numerical simulators. However, for ease in the various calculations used during well testing, the geometry of fluid flow can be classified into three forms; radial flow, linear flow and spherical flow [1,2]. In general, in the study of the mobility of fluids in porous media, a mathematical analysis is required depending on the conditions in which the filtration process takes place, the type of fluids being filtered and the geometric symmetry of the flow that is created [3,4]. The use of the production decline curves analyses is a method which serves both for the calculation of oil and gas reserves in underground reservoirs, but also for the prediction of future productivity from these reservoirs [5]. Also, another important impact is on the evaluation and audit of the various investments that are made during the exploitation and production of an oil field [6,7]. Based on this, physical treatment and mathematical analysis step by step for each decline taken into consideration [5,8]; exponential decline, harmonic decline and hyperbolic decline, which is presented in the sexion as following, is important not only in the mathematical verification of every formula taken into consideration and analyzed, but also in determining of the decline rate in some methods, something which helps in the truth of many applications carried out in reservoir engineering related to decline and the stages in which the production of an oil field has passed [9,10].

II. Methodology

Based on the proposal of Arps, that rate curves as a function of time can be expressed mathematically by one of the equations of the hyperbolic family, it is concluded that any type of curve will have a different curvature depending on the different parameters between them [2,3,5]. The theory on which the analysis of the decline curves is based begins with the nominal (instantaneous) decline rate, which is determined by the following formula [3,7]:

$$D = -\frac{d(\ln q)}{dt} = -\frac{1}{q} * \frac{dq}{dt} \tag{1}$$

Having always a decline rate D with a positive value, this is the reason for placing the (-) sign in the above expression since $\frac{dq}{dt}$ have opposite sign. Also, the decline rate can be represented by the rate q and the exponential constant b, as following:

$$D = k * q^b \tag{2}$$

$$k = \frac{D_i}{{q_i}^b} * D_i \rightarrow initial\ conditions$$

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Since for the exponential decline, the coefficient b is b=0, then the decline rate for the exponential decline will be as following:

$$D = \frac{D_i}{{q_i}^0} * q^0 = D_i$$

$$D = k * q^b = -\frac{1}{q} * \frac{dq}{dt} \rightarrow \int_0^t D * dt = \int_{q_i}^{q_t} -\frac{dq}{dt} \leftrightarrow (D * t) \Big/_0^t = -(\ln q) \Big/_{q_i}^{q_t}$$

$$D * t = \ln(q_i - q_t) \rightarrow t * D = \ln\frac{q_i}{q_t} \rightarrow -D * t = ln\frac{q_t}{q_i} \rightarrow q_t = q_i * e^{-D*t}$$

$$N_P = \int_0^t q_t * D * dt = \int_0^t q_i * e^{-D*t} \rightarrow N_P = q_i * \int_0^t e^{-D*t} * d(-D * t) = -\frac{q_i}{D}(e^{-D*t}) \Big/_0^1$$

$$N_P = -\frac{q_i}{D} * (e^{-D*t} - e^0) \rightarrow N_P = -\frac{q_i}{D} * (e^{-D*t} - 1)$$

$$\boldsymbol{N_P = \frac{q_i}{D} * (1 - e^{-D*t})} \quad (3)$$

Since $q_t = q_i * e^{-D*t}$ (4) Eq: (3) will be:

$$N_P = \frac{1}{D} * (q_i - q_i * e^{-D*t}) \leftrightarrow N_P = \frac{1}{D} * (q_i - q_t)$$

$$N_P = \frac{q_i}{D} * (1 - e^{-D*t}) \; or \; N_P = \frac{q_i - q_t}{D} \quad (5)$$

Based on equation 4 and by logarithmizing it on both sides, the following expression is obtained

$$q = q_i * e^{-D*t}$$

$$\ln(q) = \ln(q_i) - D * t \rightarrow D * t = ln(q_t) - \ln(q_i) \rightarrow \boldsymbol{D = \frac{ln\frac{q_i}{q_t}}{t}} \quad (6)$$

Picking up any two points $(t_{1,}q_1)$ and $(t_{2,}q_2)$ on the straight line will allow analytical determination of value because:

$$\ln(q_1) = \ln(q_i) - D * t_1$$

$$\ln(q_2) = \ln(q_i) - D * t_1$$

$$-D = -\frac{\ln q_1 - \ln q_i}{t_1} \rightarrow D = -\frac{(ln\, q_1 - ln\, q_i)}{t_1} \rightarrow D = \frac{ln\frac{q_i}{q_1}}{t_1}$$

$$ln\, q_i = ln\, q_1 + D * t_1$$

$$\ln q_2 = ln\, q_1 + D * t_1 - D * t_2$$

$$\ln q_2 - ln\, q_1 = D * (t_1 - t_2)$$

$$\boldsymbol{D = ln\frac{q_2}{q_1} * \frac{1}{(t_1 - t_2)}} \quad (7)$$

If production rate and cumulative production data are available, the D-value can be obtained based on the slope of the straight line on N_P versus q plot. In fact, rearranging equation:

$$N_P = \frac{q_i - q_t}{D} \text{ or } N_P = \frac{1}{D} * (q_i - q_t)$$

$$N_P = \frac{q_i - q_t}{D} \rightarrow q_t = q_i - N_P * D \rightarrow \boldsymbol{D = \frac{q_i - q_t}{N_P}} \quad (8)$$

Picking up any two points, (N_{P_1}, q_1) and (N_{P_2}, q_2), on the straight line will allow analytical determination of D-value as following:

$$q_1 = q_i - D * N_{P_1}$$

$$q_2 = q_i - D * N_{P_2}$$

$$q_i = q_1 + D * N_{P_1}$$

$$q_2 = q_i + D * N_{P_1} - D * N_{P_2}$$

$$q_1 - q_2 = D * N_{P_2} - D * N_{P_1}$$

$$q_1 - q_2 = D * (N_{P_2} - N_{P_1})$$

$$\boldsymbol{D} = \frac{\boldsymbol{q_1 - q_2}}{\boldsymbol{N_{P_2} - N_{P_1}}} \quad (9)$$

All the above formulas mathematically analyzed step by step to derive the final results for the decline rate by combining and replacing all the parameters that take part in them, it is possible to calculate the decline rate for the exponential decline. Based on this, the harmonic decline have been analyzed and considering that b=1 the calculation of decline rate is obtained as following:

$D = k * q^b$ where $k = \frac{D_i}{{q_i}^b} = \frac{D_i}{q_i}$

$$D = \frac{D_i}{q_i} * q^b = -\frac{1}{q} * \frac{dq}{dt} \rightarrow \int_0^t \frac{D_i}{q_i} * dt = \int_{q_i}^q -\frac{dq}{q^2} \rightarrow \frac{D_i}{q_i} * t = \int_{q_i}^q -q^2 * dq$$

$$\frac{D_i}{q_i} * t = -\left(\frac{q^{-2+1}}{-2+1}\right) \Big/_{q_i}^{q} \rightarrow \frac{D_i}{q_i} * t = -\left(-\frac{1}{q}\right) \Big/_{q_i}^{q} \rightarrow \frac{D_i}{q_i} * t = -\left(-\frac{1}{q} + \frac{1}{q_i}\right)$$

$$\frac{D_i}{q_i} * t = \frac{1}{q} - \frac{1}{q_i} \rightarrow \frac{D_i * t}{q_i} + \frac{1}{q_i} = \frac{1}{q} \rightarrow \frac{D_i * t + 1}{q_i} = \frac{1}{q} \rightarrow \boldsymbol{q} = \frac{\boldsymbol{q_i}}{\boldsymbol{D_i * t + 1}} \quad (10)$$

Cumulative production is defined as following:

$$N_P = \int_0^t q * dt = N_P = \int_0^t \frac{q_i}{D_i * t + 1} * dt \rightarrow N_P = q_i * \int_0^t \frac{1}{D_i * t + 1} * dt$$

$$N_P = \frac{q_i}{D_i} * \int_0^t \frac{d(1 + D_i * t)}{1 + D_i * t} = \frac{q_i}{D_i} * [\ln(1 + D_i * t)] \Big/_0^t$$

$$\frac{q_i}{D_i} * [\ln(1 + D_i * t)] \rightarrow \boldsymbol{N_P} = \frac{\boldsymbol{q_i}}{\boldsymbol{D_i}} * \boldsymbol{ln(1 + D_i * t)} \quad (11)$$

By combining Eq (10) and (11) the expression as following is obtained:

$\mathrm{q} = \frac{\mathrm{q_i}}{\mathrm{D_i * t + 1}}$ and $\mathrm{N_P} = \frac{\mathrm{q_i}}{\mathrm{D_i}} * \ln(1 + \mathrm{D_i} * \mathrm{t})$

$$q = \frac{q_i}{D_i * t + 1} \rightarrow q + q * D_i * t = q_i \rightarrow D_i = \frac{q_i - q}{q * t}$$

$$N_P = \frac{q_i}{D_i} * ln\left(1 + \frac{q_i - q}{q * t} * t\right) = \frac{q_i}{D_i} * ln\left(1 + \frac{q_i - q}{q}\right)$$

$$N_P = \frac{q_i}{D_i} * ln\frac{q_i}{q} \leftrightarrow N_P = \frac{q_i}{D_i} * [\ln(q_i) - \ln(q)] \rightarrow \boldsymbol{D_i} = \frac{\boldsymbol{q * [\ln(q_i) - \ln(q)]}}{\boldsymbol{N_P}} \quad (12)$$

III. Results And Discussion

All the above formulas that express the calculation of the decline rate, both for the exponential decline and for the harmonic decline, have been calculated by mathematically analyzing step by step all the parameters that take part in them, the combinations, substitutions and the physical concept of the relationship between theirs. Another mathematical method that can be used to calculate the coefficient of decline for both exponential and harmonic decline is the least squares method, which is illustrated as following [5,8] :

Let’s calculate the decline rate again using the least squares method. Starting from equation $D = \frac{ln\frac{q_i}{q_1}}{t_1}$ the expressions as following are obtained:

$$D = \frac{ln\frac{q_i}{q_t}}{t} \rightarrow \ D = \gamma;\ ln\frac{q_i}{q_t} = a;\ t = x$$

$\gamma = \frac{a}{\times}$ Knowing that the minimum is reached when $S^*(a) = 0$ the equation as following are obtained:

$$\frac{\partial S}{\partial a} = 0 \rightarrow \frac{\partial}{\partial a}\left[\frac{a}{\times} - \gamma\right]^2 = 0 = 2 * \left(\frac{a}{\times} - \gamma\right) * \frac{1}{\times} = 0$$

$$\frac{\partial S}{\partial a} = 0 = 2 * \sum_{i=1}^{n} \frac{a}{\times^2} - \sum_{i=1}^{n} \frac{\gamma}{\times} = 0 \rightarrow \sum_{i=1}^{n} \frac{a}{\times^2} - \sum_{i=1}^{n} \frac{\gamma}{\times} = 0$$

$$\sum_{i=1}^{n} \frac{a}{\times^2} = \sum_{i=1}^{n} \frac{\gamma}{\times} \rightarrow \gamma = \frac{\sum_{i=1}^{n} \times * a}{\sum_{i=1}^{n} \times^2} = \frac{\sum_{i=1}^{n} t * ln\frac{q_i}{q_t}}{\sum_{i=1}^{n} t^2}$$

$$\mathbf{D} = \frac{\sum_{i=1}^{n} t * \ln\frac{q_i}{q_t}}{\sum_{i=1}^{n} t^2} \quad (13)$$

$$D = \frac{q_i - q_t}{N_P} \rightarrow q_t = q_i - N_P * D$$

$$q = \gamma;\ q_i = a_1;\ N_P = \times;\ D = a_2$$

$$\gamma = a_1 - a_2 * \times$$

Knowing that the minimum conditions are $\frac{\partial S}{\partial a_1} = 0$ and $\frac{\partial S}{\partial a_2}$ will have:

$$\frac{\partial S}{\partial a_1} = 0 = \frac{\partial}{\partial a_1}[(a_1 - a_2 * \times) - \gamma]^2 = 0$$

$$\frac{\partial S}{\partial a_2} = 0 = \frac{\partial}{\partial a_2}[(a_1 - a_2 * \times) - \gamma]^2 = 0$$

$$\frac{\partial S}{\partial a_1} = 0 = 2 * [(a_1 - a_2 * \times) - \gamma] * 1 = 0$$

$$\frac{\partial S}{\partial a_2} = 0 = 2 * [(a_1 - a_2 * \times) - \gamma] * \times = 0$$

$$n * a_1 - a_2 \sum_{i=1}^{n} \times = \sum_{i=1}^{n} \gamma$$

$$a_1 * \sum_{i=1}^{n} \times - a_2 \sum_{i=1}^{n} \times^2 = \sum_{i=1}^{n} \times * \gamma$$

$$\begin{vmatrix} n & \sum_{i=1}^{n} \times \\ \sum_{i=1}^{n} \times & \sum_{i=1}^{n} \times^2 \end{vmatrix} \begin{vmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_2 \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{n} \gamma \\ \sum_{i=1}^{n} \times * \gamma \end{vmatrix}$$

$$a_1 = \frac{\sum_{i=1}^{n} \gamma * \sum_{i=1}^{n} \times^2 - \sum_{i=1}^{n} \times * \gamma *}{n * \sum_{i=1}^{n} \times^2 - (\sum_{i=1}^{n} \times^2)}$$

$$a_2 = \frac{n * \sum_{i=1}^{n} \times * \gamma - \sum_{i=1}^{n} \times * \sum_{i=1}^{n} \gamma}{n * \sum_{i=1}^{n} \times^2 - (\sum_{i=1}^{n} \times)^2}$$

Since: $\gamma = q_1$; $a_1 = q_i$; $\times = N_P$ dhe $a_2 = D$ will have:

$$\mathbf{D} = \frac{n * \sum_{i=1}^{n} (q * N_P) - \sum_{i=1}^{n} N_P * \sum_{i=1}^{n} q}{n * \sum_{i=1}^{n} (N_P)^2 - (\sum_{i=1}^{n} N_P)^2} \quad (14)$$

The decline rate presented in Eq. 13 and Eq. 14 is the decline rate in the case of exponential decline taking into account parameters such as initial rate, cumulative production for different years and the time for which the calculation was made. Based on the above methodology, taking into account the harmonic decline and other parameters that are combined and replaced in the relevant equations, the least squares method is used to calculate the decline rate as following:

$$q = \frac{q_i}{D_i * t + 1} \rightarrow q + q * D_i * t = q_i \rightarrow D_i = \frac{q_i - q}{q * t}$$

$$D_i = y;\ q_i - q = a;\ q * t = x$$

$$y = \frac{a}{x}$$

$$\frac{\partial S}{\partial a} = 0 \rightarrow \frac{\partial}{\partial a}\left[\frac{a}{x} - y\right]^2 = 0 = 2 * \left(\frac{a}{x} - y\right) * \frac{1}{x} = 0$$

$$\frac{\partial S}{\partial a} = 0 = 2 * \sum_{i=1}^{n} \frac{a}{\times^2} - \sum_{i=1}^{n} \frac{\gamma}{\times} = 0 \rightarrow \sum_{i=1}^{n} \frac{a}{\times^2} - \sum_{i=1}^{n} \frac{\gamma}{\times} = 0$$

$$\sum_{i=1}^{n}\frac{a}{\times^2}=\sum_{i=1}^{n}\frac{\gamma}{\times}\to\gamma=\frac{\sum_{i=1}^{n}\times * a}{\sum_{i=1}^{n}\times^2}\to D_i=\frac{\sum_{i=1}^{n}[(q_i-q)*(q*t)]}{\sum_{i=1}^{n}[q^2t^2]}=\frac{\sum_{i=1}^{n}[q_i*q*t-q^2]}{\sum_{i=1}^{n}[q^2t^2]}$$

$$\boldsymbol{D_i}=\frac{\sum_{i=1}^{n}\frac{q*t}{q}-\sum_{i=1}^{n}t}{\sum_{i=1}^{n}t^2} \tag{15}$$

All the calculations and mathematical combinations above, the application of rules and methods both in the physical context and in that mathematics step by step, make it possible not only to calculate the decline rate for each decline taken into consideration, the exponential decline and the harmonic decline, but also a verification of each formula by which the decline rate is determined, which will help us in practical applications related to the production of fields of oil and gas.

IV. Conclusion

Decline curve analysis can be applied to almost any manufacturing operation hydrocarbons, which is applied both in technical assistance for the forecasting and future development of underground oil and gas reservoirs, but also in a forecast and economic analysis in the various investments that will be developed in the field including here equipment and facilities such as pipelines, plants, and treating facilities. All the above formulas that express the calculation of the decline rate, both for the exponential decline and for the harmonic decline, have been calculated by mathematically analyzing step by step all the parameters that take part in them, the combinations, substitutions and the physical concept of the relationship between theirs, which help us in the truth of the value of the decline rate during various applications both in the site and in the relevant software for the calculation and prediction of hydrocarbon reserves.

V. Nomenclature

$\mathbf{q_i} \to initial\ oil\ flow\ rate$
$\mathbf{q_t} \to oil\ flow\ rate\ at\ time\ t$
$\boldsymbol{t} \to time, year$
$\mathbf{D} \to nominal\ decline\ rate$
$\boldsymbol{D_i} \to initial\ decline\ rate$
$\boldsymbol{N_{P\,\to}}\ cumulative\ production$

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