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Transient Analysis of a Queueing System with Feedback, Catastrophes and State-Dependent Environmental Change Parameter

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Abstract: - This paper presents atransient analysis of a finite capacity queueing system with feedback, catastrophes and state-dependent environmental change parameter. The Poisson rate at which the system moves from environmental changing state F to E increases or decreases as the number of customers in the queue. Also, at some random times, the number of customers is immediately reset to zero whenever a catastrophe occurs at the system. Transient solution is obtained by using the technique of probability generating function. The Steady state solution of the model is obtained by using the property of Laplace transform. Further we discussed several particular cases of the queueing modelwith and without feedback and catastrophes.

Keywords: Transient analysis, Feedback, Catastrophes, Environmental change, Finite capacity, Probability generating function.

1. INTRODUCTION:

Analytical results for the transient behavior of a queueing models are not as prevalent as those for steady-state conditions. Steady-state measures fail to provide insights into the system's transient behavior, which is crucial for understanding its dynamics over finite periods. Among various methods available, the probability generating function technique is particularly effective for deriving transient solutions. However, even for a simple M/M/1/N queue, obtaining analytical expressions for transient behavior proves to be quite challenging. In this context, we have successfully derived the transient solution for a finite capacity queueing system characterized by a state-dependent environmental change parameter along with the feedback and catastrophic effects.

Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. This usually happens because of non-satisfactory quality of service. Also,in queueing systems numerous authors have introduced the concept of catastrophes. In particular, birth and death models have been extended to include the assumption that the number of customers is instantly reset to zero at certain random times. These catastrophes occur at the service facility as a Poisson process with a specified rateξ. When a catastrophe happens, all customers present are immediately removed, the server is temporarily inactivated, and it becomes available for service only when a new customer arrives in the system.

The queueing system with catastrophes was first examined by Krishna Kumar and Arivudainambi[9] in 2000, followed by Crescenzo et al.[5] in 2003, who derived the transient probabilities for the M/M/1 queueing model with catastrophes. Jain and Kanethia [8] explored the transient analysis of a queueing model with state dependent environmental change parameter and catastrophes.Kumar, D.[11] furtherexplores a queueing system that incorporates catastrophes, state-dependent input parameters, and environmental changes to assess their impact on the system behavior.Takacs [15], in his interesting paper has introduced a queue with feedback. Sharma and Kumar [12] considered a Markovian feedback queue with retention of reneged customers in which the inter-arrival and service times follow exponential distribution.

In this paper, we introduce an additional factor of environmental change, whereby the changes in the environment influence the state of the queueing system. Specifically, the state of the queueing system is a function of these environmental change factors. When the environment shifts, the system

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transitions from environmental state E to state F at a certain rate β, while it moves from state F to state E at a rate ofbn.

The primary objective of this paper is to develop a finite capacity queueing system that incorporates the effects of feedback, catastrophes and a state-dependent environmental change parameter. In Section 2, we outline the assumptions and definitions related to the model. Section 3 provides a detailed analysis of the queueing model, while specific cases are explored in Section 4. Finally, the steady-state results are mentioned in sections 5 and 6 along with the application of the model.

2. ASSUMPTIONS AND DEFINITIONS OF THE MODEL:

a) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non-homogeneous i.e., there may exists two arrival rates, namely λ_1 and 0 of which only one is operative at any instant.

b) The customers are served one by one at the single service channel. The service time is exponentially distributed. Further, corresponding to arrival rate λ_1 the Poisson service rate is μ_1 and the service rate corresponding to the arrival rate 0 is μ_2. The state of the system when operating with arrival rate λ_1 and service rate μ_1 is designated as E whereas the other with arrival rate 0 and service rate μ_2 is designated as F.

c) In state E after obtaininga service, an unsatisfied customer may rejoin the queue for receiving another service with probability 1-p (=q), referred to as "feedback"or they can choose to leave the system permanently with probability p, p+q=1. There is no feedback in state F of the system.

d) The Poisson rate at which the system goes from environmental state E to F is denoted by β. Also the Poisson rate b_n, at which the system moves from environmental state F to E tends to increase or decrease according as the numbers in the queue (say n) increase or decrease from some fixed number (say N). We therefore define,

$$b_n = \alpha\left[1+\varepsilon\left(n-N\right)\right] \text{ with } n \geq N - \frac{1}{\varepsilon}$$

$$\text{and } 0 \leq N - \frac{1}{\varepsilon} \leq n \leq M$$

Where M denotes the size of the waiting space and ε is a positive number such that $\varepsilon \geq \frac{1}{N}$. This restriction on M is necessary to avoid a negative value of bn. When n=N or ε =0, bn gives the normal rate as α.

e) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty. Simultaneously, the system becomes ready to accept the new customers.

f) The queue discipline is first-come-first-served.

g) The capacity of the queueing system is restricted to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue and will be considered lost for the system.

Define,

P_n (t) = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

Q_n(t) =Joint probability that at time t the system is in state F and nunits are in the queue, including the one in service.

$R_n(t)$= The probability that at time t there are n units in the queue,including the one in service.

Obviously,

$$R_n(t) = P_n(t) + Q_n(t)$$

Let us measure time t from an instant when there are zero customers in the queue and the system is in the environmental state E so that the initial conditions associated with $P_n(t)$ and $Q_n(t)$ becomes,

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$Q_n(0) = 0$; for all n.

3. FORMULATION OF THE QUEUEING MODEL AND TRANSIENT ANALYSIS:

The differential-difference equations governing the system are.

$$\frac{d}{dt}P_0(t) = -(\lambda_1+\beta+\xi)P_0(t) + \mu_1 p P_1(t) + b_0 Q_0(t) + \xi\sum_{n=0}^{M} P_n(t) \; ; n = 0 \tag{1}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1+\mu_1 p+\beta+\xi)P_n(t) + \mu_1 p P_{n+1}(t) + \lambda_1 P_{n-1}(t) + b_n Q_n(t) \; ; \; 0 < n < M \tag{2}$$

$$\frac{d}{dt}P_M(t) = -(\mu_1 p+\beta+\xi)P_M(t) + \lambda_1 P_{M-1}(t) + b_M Q_M(t) \; ; n = M \tag{3}$$

$$\frac{d}{dt}Q_0(t) = -(b_0+\xi)Q_0(t) + \mu_2 Q_1(t) + \beta P_0(t) + \xi\sum_{n=0}^{M} Q_n(t) \; ; n = 0 \tag{4}$$

$$\frac{d}{dt}Q_n(t) = -(\mu_2+b_n+\xi)\,Q_n(t) + \mu_2\,Q_{n+1}(t) + \beta P_n(t) \; ; 0 < n < M \tag{5}$$

$$\frac{d}{dt}Q_M(t) = -(\mu_2+b_M+\xi)Q_M(t) + \beta P_M(t) \; ; \quad n = M \tag{6}$$

Define, the Laplace Transform as

$$\text{L.T. } [f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s) \tag{7}$$

Now, taking the Laplace transforms of equations (1)–(6) and using the initial conditions, we get

$$(s+\lambda_1+\beta+\xi)\overline{P}_0(s) - 1 = \mu_1 p\,\overline{P}_1(s) + b_0\,\overline{Q}_0(s) + \xi\sum_{n=0}^{M}\overline{P}_n(s) \tag{8}$$

$$(s+\lambda_1+\mu_1 p+\beta+\xi)\overline{P}_n(s) = \mu_1 p\,\overline{P}_{n+1}(s) + \lambda_1\,\overline{P}_{n-1}(s) + b_n\,\overline{Q}_n(t) \tag{9}$$

$$(s+\mu_1 p+\beta+\xi)\overline{P}_M(s) = \lambda_1\,\overline{P}_{M-1}(s) + b_M\,\overline{Q}_M(s) \tag{10}$$

$$(s+b_0+\xi)\overline{Q}_0(s) = \mu_2\,\overline{Q}_1(s) + \beta\,\overline{P}_0(s) + \xi\sum_{n=0}^{M}\overline{Q}_n(s) \tag{11}$$

$$(s+\mu_2+b_n+\xi)\overline{Q}_n(s) = \mu_2\,\overline{Q}_{n+1}(s) + \beta\,\overline{P}_n(s) \tag{12}$$

$$(s+\mu_2+b_M+\xi)\overline{Q}_M(s)=\beta\,\overline{P}_M(s) \tag{13}$$

Define, the probability generating functions

$$P(z,s)=\sum_{n=0}^{M}\overline{P}_n(s)z^n \tag{14}$$

$$Q(z,s)=\sum_{n=0}^{M}\overline{Q}_n(s)z^n \tag{15}$$

$$R(z,s)=\sum_{n=0}^{M}\overline{R}_n(s)z^n \tag{16}$$

where

$$\overline{R}_n(s)=\overline{P}_n(s)+\overline{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$\alpha\varepsilon z^2 Q'(z,s)+\alpha(1-\varepsilon N)zQ(z,s)+\left[\lambda_1 z^2-z\{s+\lambda_1+\mu_1 p+\xi+\beta\}+\mu_1 p\right]P(z,s)$$
$$=\lambda_1 z^{M+1}(z-1)\overline{P}_M(s)+\mu_1 p(1-z)\overline{P}_0(s)-z-\xi z\sum_{n=0}^{M}\overline{P}_n(s) \tag{17}$$

Similarly, from equations (11)–(13) and using (14)–(16), we have

$$\alpha\varepsilon z^2 Q'(z,s)+\left[z\{s+\mu_2+\xi+\alpha(1-\varepsilon N)\}-\mu_2\right]Q(z,s)-\beta\, zP(z,s)$$
$$=\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s) \tag{18}$$

Eliminating P(z,s) from equations (17) and (18), we have

$$Q'(z,s)+\frac{\eta_1(z)}{\eta_2(z)}Q(z,s)=\frac{1}{\eta_2(z)}\left[z_1+z_2\overline{Q}_0(s)+z_3\overline{P}_0(s)+z_4\overline{P}_M(s)+z_5\sum_{n=0}^{M}\overline{P}_n(s)+z_6\sum_{n=0}^{M}\overline{Q}_n(s)\right] \tag{19}$$

where

$$\eta_1(z)=\left[a_2\lambda_1 z^3+\{\alpha\beta(1-\varepsilon N)-a_2a_3\}z^2+(a_2\mu_1 p+a_3\mu_2)z-\mu_1 p\mu_2\right]$$

$$\eta_2(z)=\alpha\varepsilon z^2\left[\lambda_1 z^2+\mu_1 p-a_1 z\right]$$

$$z_1=-\beta z^2$$

$$z_2=\mu_2(z-1)\left[\lambda_1 z^2-a_3 z+\mu_1 p\right]$$

$$z_3=\beta\mu_1 pz(1-z)$$

$$z_4=\beta\lambda_1 z^{M+2}(z-1)$$

$$z_5=-\beta\xi z^2$$

$$z_6=\xi z\left[\lambda_1 z^2-a_3 z+\mu_1 p\right]$$

$$a_1 = [s + \mu_1 p + \lambda_1 + \xi]$$

$$a_2 = [s + \mu_2 + \xi + \alpha(1 - \varepsilon N)]$$

$$a_3 = [s + \lambda_1 + \mu_1 p + \beta + \xi]$$

In equation (19), the co-efficient of Q(z,s) can be re-written as

$$\frac{\eta_1(z)}{\eta_2(z)} = \frac{A_1}{z} + \frac{B_1}{z^2} + \frac{C}{2\lambda_1}\left[\frac{2\lambda_1 z - a_1}{\lambda_1 z^2 - a_1 z + \mu_1 p}\right] + \frac{D_1}{X^2(z) - a^2} \tag{20}$$

where

$$A_1 = \frac{a_2}{\alpha\varepsilon} + A$$

$$A = \frac{\mu_1 p a_4 - a_1 a_5}{\alpha\varepsilon\mu_1^{\,2} p}$$

$$B_1 = [\alpha\beta(1 - \varepsilon N) - a_2 a_3 + a_1 a_2]\frac{1}{\lambda_1 \alpha\varepsilon} + \frac{B}{\alpha\varepsilon}$$

$$B = -\frac{a_5}{\mu_1 p}$$

$$C = \frac{\lambda_1(a_1 a_5 - \mu_1 p a_4)}{\alpha\varepsilon\mu_1^{\,2} p}$$

$$D_1 = D + \frac{C a_1}{2\lambda_1}$$

$$D = a_1(\mu_1 p\, a_4 - a_1 a_5)\frac{1}{\mu_1^{\,2} p\, \alpha\, \varepsilon} + \frac{\lambda_1\, a_5}{\mu_1 p\, \alpha\, \varepsilon}$$

$$X(z) = \left[z\lambda_1^{1/2} - \frac{a_1}{2\lambda_1^{1/2}} \right]$$

$$a = \left[\frac{a_1^{\,2}}{4\lambda_1} - \mu_1 p\right]^{1/2}$$

$$a_4 = a_3\mu_2 + \frac{a_1}{\lambda_1}[\alpha\beta(1 - \varepsilon N) - a_2 a_3 + a_1 a_2]$$

$$a_5 = \frac{\mu_1 p}{\lambda_1}[\alpha\beta(1 - \varepsilon N) - a_2 a_3 + a_1 a_2] + \mu_1 p \mu_2$$

Using equation (20) in equation (19) and integrating (19) w. r. t. z, we have

$$Q(z,s) = \frac{L_1(z) + L_2(z)\overline{Q}_0(s) + L_3(z)\overline{P}_0(s) + L_4(z)\overline{P}_M(s) + L_5(z)\sum_{n=0}^{M}\overline{P}_n(s) + L_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L(z)} \tag{21}$$

where

$$L(z)=z^{A_1}\left(\lambda_1 z^2-a_1 z+\mu_1 p\right)^{C/2\lambda_1}\left[\frac{X(z)-a}{X(z)+a}\right]^{D_1/\ 2a}\cdot\exp\left(-\frac{B_1}{z}\right)$$

$$L_j(z)=\int_0^z \frac{z_j}{\eta_2(z)}L(z)\,dz\ ;\qquad j=1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

Again eliminating $Q'(z,s)$ from equations (17) and (18), and using Q(z,s) from equation (21), we have

$$P(z,s)=\frac{L_7(z)+\overline{Q}_0(s)L_8(z)+\overline{P}_0(s)L_9(z)+\overline{P}_M(s)L_{10}(z)+\sum_{n=0}^{M}\overline{P}_n(s)L_{11}(z)+\sum_{n=0}^{M}\overline{Q}_n(s)L_{12}(z)}{B(z)L(z)}\tag{22}$$

where

$$L_7(z)=L_1(z)\,g(z)-z\,L(z)$$

$$L_8(z)=L_2(z)\,g(z)-\mu_2(z-1)\,L(z)$$

$$L_9(z)=L_3(z)\ g(z)-\mu_1 p(z-1)\,L(z)$$

$$L_{10}(z)=L_4(z)\ g(z)+\lambda_1 z^{M+1}(z-1)\,L(z)$$

$$L_{11}(z)=L_5(z)\ g(z)-\xi z\ L(z)$$

$$L_{12}(z)=L_6(z)\ g(z)-\xi z\ L(z)$$

$$g\ (z)=(s+\xi)z+\mu_2(z-1)$$

$$B(z)=\lambda_1 z^2+\mu_1 p-a_1 z$$

Adding equations (21) and (22), we have

$$R(z,s)=\frac{C_1(z)+C_2(z)\overline{Q}_0(s)+C_3(z)\overline{P}_0(s)+C_4(z)\overline{P}_M(s)+C_5(z)\sum_{n=0}^{M}\overline{P}_n(s)+C_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B(z)\,L(z)}\tag{23}$$

where

$$C_i(z)=B(z)\,L_i(z)+L_{i+6}(z)\ ;\ \ i=1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

Since, $\sum_{n=0}^{M}R_n(t)=1$

Therefore,

$$R(1,s)=\sum_{n=0}^{M}\overline{R}_n(s)=\frac{1}{s}$$

Thus equation (23) for z=1, gives

$$R(1,s)=\frac{1}{s}=\lim_{z\to 1}R(z,s)\tag{24}$$

Also

$$P(0,s)=\overline{P}_0(s)=\lim_{z\to 0} P(z,s) \tag{25}$$

and $$Q(0,s)=\overline{Q}_0(s)=\lim_{z\to 0} Q(z,s) \tag{26}$$

The equations (24), (25) and (26) on solution gives the values of $\overline{P}_0(s)$, $\overline{Q}_0(s)$, $\overline{P}_M(s)$, $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$. Let them be P_0, Q_o, P_M, $\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}Q_n$ respectively then from relation (24), we have

$$R(z,s)=\frac{C_1(z)+C_2(z)Q_0+C_3(z)P_0+C_4(z)P_M+C_5(z)\sum_{n=0}^{M}P_n+C_6(z)\sum_{n=0}^{M}Q_n}{B(z)\,L(z)} \tag{27}$$

The Laplace transform of various state probabilities for the number of customers in the queue, including the one in service can be obtained as the coefficients of different powers of z in the binomial expansion of equation (27).

Again, since

P(1,s)=Laplace transform of the probability that thesystemisinenvironmental state E.

and

Q(1,s)=Laplace transform of theprobability that thesystem is in environmental state F.

We have from equations (21) and (22) on setting z=1 and $\overline{P}_0(s)$ = P_0, $\overline{Q}_0(s)$ =Q_0, $\overline{P}_M(s)$ = P_M, $\sum_{n=0}^{M}\overline{P}_n(s)=\sum_{n=0}^{M}P_n$ and $\sum_{n=0}^{M}\overline{Q}_n(s)=\sum_{n=0}^{M}Q_n$

$$P(1,s)=\frac{L_1(1)+L_2(1)Q_0+L_3(1)P_0+L_4(1)P_M+L_5(1)\sum_{n=0}^{M}P_n+L_6(1)\sum_{n=0}^{M}Q_n}{L(1)} \tag{28}$$

$$Q(1,s)=\frac{L_7(1)+L_8(1)Q_0+L_9(1)P_0+L_{10}(1)P_M+L_{11}(1)\sum_{n=0}^{M}P_n+L_{12}(1)\sum_{n=0}^{M}Q_n}{B(1)\,L(1)} \tag{29}$$

These on inversions give the respective probabilities for the system to be in the environmental states E and F.

4. PARTICULAR CASE:

Setting ε =0 or n=N in equations (17) and (18), (i.e., when the rate of change of environmental state F to E is constant), we have

$$X_1(z)P(z,s)+X_2(z)Q(z,s)+X_3(z)=0 \tag{30}$$

$$X_4(z)P(z,s)+X_5(z)Q(z,s)+X_6(z)=0 \tag{31}$$

where

$$X_1(z)=-\left[\lambda_1 z^2-z\,(s+\lambda_1+\mu_1 p+\beta+\xi)+\mu_1 p\right]$$

$$X_2(z)=-\alpha z$$

$$X_3(z) = -\left[\mu_1 p(z-1)\,\overline{P}_0(s) + \lambda_1 z^{M+1}(1-z)\overline{P}_M(s) + z + \xi z \sum_{n=0}^{M} \overline{P}_n(s)\right]$$

$$X_4(z) = \beta z$$

$$X_5(z) = [\mu_2 - z(s + \mu_2 + \alpha + \xi)]$$

$$X_6(z) = \left[\mu_2 (z-1)\overline{Q}_0(s) + \xi z \sum_{n=0}^{M} \overline{Q}_n(s)\right]$$

From equations (30) and (31), we have

$$P(z,s) = \frac{X_2(z)X_6(z) - X_3(z)X_5(z)}{X_1(z)X_5(z) - X_2(z)X_4(z)} \tag{32}$$

$$Q(z,s) = \frac{X_4(z)X_3(z) - X_1(z)X_6(z)}{X_1(z)X_5(z) - X_2(z)X_4(z)} \tag{33}$$

Thus, we have

$$R(z,s) = \frac{\begin{array}{l}\mu_2(z-1)[X_2(z) - X_1(z)]\overline{Q}_0(s) + \xi z \sum_{n=0}^{M} \overline{Q}_n(s)\,[X_2(z) - X_1(z)] + \mu_1 p(1-z) \\ [X_4(z) - X_5(z)]\overline{P}_0(s) + \lambda_1 z^{M+1}[X_5(z) - X_4(z)]\,(1-z)\overline{P}_M(s) + \\ z[X_5(z) - X_4(z)] + \xi z \sum_{n=0}^{M} \overline{P}_n(s)\,[X_5(z) - X_4(z)]\end{array}}{-z^2 s^2 + s X_7(z) + (1-z)X_8(z) - z^2 \xi(\alpha + \beta + \xi)} \tag{34}$$

where

$$X_7(z) = \lambda_1 z^3 - z^2(\lambda_1 + \mu_1 p + \mu_2 + \alpha + \beta + 2\xi) + z(\mu_1 p + \mu_2)$$

$$X_8(z) = -z^2 \lambda_1(\alpha + \mu_2 + \xi) + z\,[\alpha \mu_1 p + \mu_2(\lambda_1 + \mu_1 p + \beta + \xi)] - \mu_1 p \mu_2 .$$

$$P(1,s) = \sum_{n=0}^{M} \overline{P}_n(s) = \frac{s + \alpha + \xi}{s(s + \alpha + \beta + \xi)}$$

$$Q(1,s) = \sum_{n=0}^{M} \overline{Q}_n(s) = \frac{\beta}{s(s + \alpha + \beta + \xi)}$$

and

$$\sum_{n=0}^{M} \overline{R}_n(s) = \sum_{n=0}^{M} \overline{P}_n(s) + \sum_{n=0}^{M} \overline{Q}_n(s) = \frac{1}{s}$$

Relation (34) is a polynomial in z and holds for all values of z, including the three zeros of the denominator. Hence $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ are obtained by setting the numerator equal to zero and substituting the three zeros, α_1, α_2 and α_3 (say) of the denominator (at each of which the numerator must vanish).

Case 1: Let $\alpha \to \infty$, $\beta \to 0$ and setting $\mu_1 = \mu_2 = \mu$ (say) in relation (34), we have

$$r(z,s) = \frac{(1-z)\,\mu\,\overline{R}_0(s) - (1-z)\lambda_1 z^{M+1}\,\overline{P}_M(s) - z - \xi z/s}{\lambda_1 z^2 - z(s + \lambda_1 + \mu + \xi) + \mu} \tag{35}$$

where

$$\overline{R}_0(s) = \overline{P}_0(s) + \overline{Q}_0(s)$$

$$r(z,s) = \lim_{\beta \to 0}\left[\lim_{\alpha \to \infty} R(z,s)\right]$$

Relation (35) is a polynomial in z and exists for all values of z, including the two zeros of the denominator. Hence, $\overline{R}_0(s)$ and $\overline{P}_M(s)$ can be evaluated as before. **Case 2:** In relation (34), if p=1, i.e. when there are no feedback customers. The model reduces to one which is studied by Jain and Kanethia [8].

5. STEADY STATE RESULTS:

This can be obtained by the well-known property of the Laplace transform given below:

$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, \bar{f}(s)$,If the limit on the left hand side exists.

Thus if $R(z) = \sum_{n=0}^{M} R_n z^n$

where,

$$R_n = \lim_{s \to 0} s \overline{R}_n(s)$$

Then $R(z) = \lim_{s \to 0} s\ R\ (z,s)$

By employing this property, from equation (23)we have

$$R(z) = \frac{Q_0 N_1(z) + N_2(z)P_0 + N_3(z)P_M + N_4(z)\sum_{n=0}^{M} P_n + N_5(z)\sum_{n=0}^{M} Q_n + N}{B_1(z)\,N(z)} \tag{36}$$

where,

$$N_i(z) = B_1(z)L'_{i+1}(z) + L'_{i+7}(z)\Big|_{s=0} \quad ;\ i=1, 2, 3, 4, 5.$$

$$B_1(z) = B(z)\big|_{s=0}$$

$$L'_j(z) = \int \left[\frac{z_j}{\eta_2(z)} L(z)\right]_{\epsilon=0} dz \quad ;\ j=2, 3, 4, 5, 6.$$

$$L'_k(z) = L_k(z)\big|_{s=0} \quad ;\ k=8, 9, 10, 11, 12.$$

$$N(z) = L(z)\big|_{s=0}$$

N' = the constant of integration.

The unknown quantities Q_0, P_0, P_M, $\sum_{n=0}^{M} P_n$ and $\sum_{n=0}^{M} Q_n$ can be evaluated as before.

Particular case:

Relation (34), on applying the theory of Laplace transforms gives

$$R(z)=\frac{\begin{array}{l}\mu_2(1-z)\{\alpha z+z(\lambda_1+\mu_1 p+\beta+\xi)-\lambda_1 z^2-\mu_1 p\}Q_0+\\ \mu_1 p(1-z)[\beta z-\{\mu_2-z(\mu_2+\alpha+\xi)\}]P_0+\lambda_1 z^{M+1}(1-z)\{\mu_2-z(\mu_2+\alpha+\xi)-\beta z\}P_M+\\ \xi z/(\alpha+\beta+\xi)[\beta\{\lambda_1 z^2-z(\lambda_1+\mu_1 p+\alpha+\beta+\xi)+\mu_1 p\}+(\alpha+\xi)\{\mu_2-z(\mu_2+\alpha+\beta+\xi)\}]\end{array}}{\begin{array}{l}z^3\lambda_1(\mu_2+\alpha+\xi)-z^2[\lambda_1(\mu_2+\alpha+\xi)+\{\alpha\mu_1 p+\mu_2(\lambda_1+\mu_1 p+\beta+\xi)\}+\xi(\alpha+\beta+\xi)]\\ +z[\{\alpha\mu_1 p+\mu_2(\lambda_1+\mu_1 p+\beta+\xi)\}+\mu_1 p\mu_2]-\mu_1 p\mu_2\end{array}}$$

(37)

or, we can write

$$R(z)=\frac{T(z)Q_0+N(z)P_0+L(z)P_M+M(z)}{K(z)} \qquad (38)$$

Where T(z), N(z) and L(z) are the co-efficient of Q_0, P_0 and P_M respectively in the numerator of equation (37) and K(z) is the denominator of equation (37).

Equation (38) is a polynomial in z and holds for all values of z, including three zeros of the denominator. Hence Q_0, P_0 and P_M can be obtained by setting the numerator equal to zero. Substituting the three zeros b_1, b_2 and b_3 (say) of the denominator (at each of which the numerator must vanish).

Three equations determining the constants Q_0, P_0 and P_M are

$$T(b_1)Q_0+N(b_1)P_0+L(b_1)P_M=-M(b_1) \qquad (39)$$

$$T(b_2)Q_0+N(b_2)P_0+L(b_2)P_M=-M(b_2) \qquad (40)$$

$$T(b_3)Q_0+N(b_3)P_0+L(b_3)P_M=-M(b_3) \qquad (41)$$

After solving these equations, we have

$$Q_0=\frac{-M(b_1)A_{11}+M(b_2)A_{21}-M(b_3)A_{31}}{A}$$

$$P_0=\frac{M(b_1)A_{12}-M(b_2)A_{22}+M(b_3)A_{32}}{A}$$

$$P_M=\frac{-M(b_1)A_{13}+M(b_2)A_{23}-M(b_3)A_{33}}{A}$$

where

$$A=\begin{vmatrix}T(b_1) & N(b_1) & L(b_1)\\ T(b_2) & N(b_2) & L(b_2)\\ T(b_3) & N(b_3) & L(b_3)\end{vmatrix}$$

A_{ij} is the co-factor of the $(i, j)^{th}$ element of A.

By putting the values of Q_0, P_0 and P_M in equation (38), we have

$$R(z)=\frac{\begin{array}{l}T(z)[-M(b_1)A_{11}+M(b_2)A_{21}-M(b_3)A_{31}]+N(z)[M(b_1)A_{12}-M(b_2)A_{22}+M(b_3)A_{32}]\\ +L(z)[-M(b_1)A_{13}+M(b_2)A_{23}-M(b_3)A_{33}]+A\cdot M(z)\end{array}}{A\cdot K(z)} \qquad (42)$$

Mean Queue Length:

Define,

L_q=Expected number of customers in the queue including theone in service.

Then $L_q = R'(z)\big|_{z=1}$

Therefore, from equation (42), we have

$$L_q = \frac{\begin{array}{l} K(1)[T'(1)\{-M(b_1)A_{11}+M(b_2)A_{21}-M(b_3)A_{31}\}+N'(1)\{M(b_1)A_{12}-M(b_2)A_{22}+M(b_3)A_{32}\} \\ +L'(1)\{-M(b_1)A_{13}+M(b_2)A_{23}-M(b_3)A_{33}\}+A\cdot M'(1)]-[T(1)\{-M(b_1)A_{11}+M(b_2)A_{21} \\ -M(b_3)A_{31}\}+N(1)\{M(b_1)A_{12}-M(b_2)A_{22}+M(b_3)A_{32}\}+L(1)\{-M(b_1)A_{13}+M(b_2)A_{23} \\ -M(b_3)A_{33}\}+A\cdot M(1)]\,K'(1) \end{array}}{A\cdot[K(1)]^2} \tag{43}$$

where dashes denotes the first derivative with respect to z.

Now, Relation (35) on applying the theory of Laplace transforms gives

$$r(z) = \frac{(1-z)\mu R_0 - (1-z)\lambda_1 z^{M+1} P_M - \xi z}{\lambda_1 z^2 - z(\lambda_1+\mu+\xi)+\mu} \tag{44}$$

where

$$r(z) = \lim_{s\to 0} s\, r(z,s)$$

Equation (44) is a polynomial in z and holds for all values of z, including the two zeros of the denominator. Therefore, R_0 and P_M can be found by setting the numerator equal to zero. Substituting the two zerosa_1 and a_2 (say) of the denominator (at each of which the numerator must also vanish).

If $\xi = 0$ and p=1(i.e. catastrophe and feedback arenot allowed in the system):

From equation (35), on applying the theory of Laplace transforms gives

$$r(z) = \frac{\mu R_0 - \lambda_1 z^{M+1} P_M}{\mu - \lambda_1 z} \tag{45}$$

The condition, $\lim_{z\to 1} r(z) = 1$ gives

$$\mu R_0 - \lambda_1 P_M = \mu - \lambda_1 \tag{46}$$

As r(z) is analytic, the numerator and denominator of equation (45) must vanish simultaneously for z= μ/λ_1, which is a zero of its denominator. Equating the numerator of equation (45) to zero for z= μ/λ_1 we have

$$R_0 = \rho^{-M} P_M, \qquad \rho = \lambda_1/\mu < 1 \tag{47}$$

Relation (46) and (47) gives

$$R_0 = \frac{1-\rho}{1-\rho^{M+1}}, \quad P_M = \frac{(1-\rho)\rho^M}{1-\rho^{M+1}}$$

Now, from equation (45), we have

$$r(z) = \frac{1-\rho}{1-\rho^{M+1}} \cdot \left[\frac{1-(\rho z)^{M+1}}{1-\rho z}\right] \tag{48}$$

which is a well-known result of the M/M/1 queue with finite waiting space M.

When there is an infinite waiting space, the corresponding expression for r(z) is obtained by letting M tends to infinity in equation (48), If $(\rho,|z|) < 1$.

$$r(z) = \frac{1-\rho}{1-\rho z} \tag{49}$$

which is again a well-known result of M/M/1 queue with infinite waiting space.

Steady-state probabilities of the M/M/1 Queue with Catastrophes:

Kumar and Arivudainambi [9] explores an M/M/1 queueing model that incorporates the effects of catastrophes, which are sudden events causing the queue to drop to zero i.e., all customers in the system are removed and reset to zero at once. This approach is useful for modeling systems that may experience random, complete resets, such as in certain network or service systems where failures can clear out all ongoing tasks.

When a catastrophe occurs at the service facility with rate ξ, the steady-state distribution $\{p_n;\ n \geq 0\}$ of the M/M/1 queue with catastrophes corresponds to

$$p_0 = (1-\rho)\ ;\ \ n = 0 \tag{50}$$

$$p_n = (1-\rho)\rho^n\ ;\ \ n = 1, 2, 3, \ldots\ldots \tag{51}$$

where

$$\rho = \frac{(\lambda+\mu+\xi) - \sqrt{\lambda^2+\mu^2+\xi^2+2\lambda\xi+2\mu\xi-2\lambda\mu}}{2\mu} \tag{52}$$

Thus equations (50)-(52) provide the steady- state distribution for the queueing system. Obviously, the steady state distribution exists if and only if $\rho < 1$. It is also observed that the results of equations (50)-(52) agree with the model discussed above and with Chao, X [3].

6. CONCLUSION AND APPLICATION OF THE MODEL:

In this paper, we established a queueing model and obtained the transient solution of a limited capacity queueing system with feedback, catastrophes and state-dependent environmental change parameter. Further, we have derived the steady state result and the mean queue length of the model. In particular, queues with feedback, catastrophes and environmental changing states occur in industries of production systems subject to rework, computer networks, telecommunication systems, banking sectors, hospital management, etc.

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