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Spectral Density of Cooper Pairs in Parallel Double Coupled Quantum Dots – Superconductor Josephson Junction: A Model Study

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Abstract: - The present work deals with the theoretical model analysis of spectral density of Cooper pairs in double T-shape coupled quantum dots sandwiched between two conventional superconducting leads i.e. (S-DQD-S) Josephson junction. We have considered two single level quantum dots coupled with each other in T-shape and have modelled the Hamiltonian for (S-DQD-S) junction as extended two impurity Anderson model. For this purpose, we have considered an attractive BCS-type effective interaction in superconducting lead to give rise bound Cooper pairs, coupled dots states, and also coupling between superconducting leads and one of the quantum dot state. The expression of spectral density of Cooper pairs have been obtained with the help of model Hamiltonian within BCS mean field Green's function equation of motion approach. It is pointed that the spectral density of Cooper pairs mimic the zero bias conductance and exhibit tuneable Josephson effect in such (S-DQD-S) junction. On the basis of numerical computation, it is found that the spectral density depend on binding energy of Cooper pairs in the leads, coupling of energy level of the dots, dots level energy with respect to Fermi energy, and also on the coupling parameter between superconducting leads and one of the quantum dot states in an essential way. The rich physics of spectral density in S-DQD-S junctions can provide clues to experimentalist to exhibit controllable Josephson effect leading to maximum Josephson super-current in such S-DQD-S junctions to underline potential applications in quantum devices.

Keywords: Electronic structure of Quantum dot, S-QD-S Josephson Junction, BCS Superconductors , Extended two- impurity Anderson model.

1. Introduction

In recent past advancement in materials fabrication at nano-scale, made it possible to realize superconducting quantum dot nanoscopic Josephson junctions having their potential technological applications in electronic devices. The quantum dots (QD) are semiconductor nanoscopic structures, where electrons are confined to zero dimensions and due to the quantum confinement these QD have discrete energy levels like an atom[1-3]. Therefore, the QD is a class of nano-materials, having discrete energy levels and the possibility to tune the separation between energy levels and on dot Coulomb energy by changing the size, shape and environment of the QD. In superconducting materials, the electronic states close to the Fermi level are the bound electron Cooper pairs and electrons of Cooper pair can tunnel coherently through the discrete energy levels of quantum dot serving as a barrier in (S-QD-S) junction[3-4]. The QD believed to have potential applications in quantum devices like quantum computers and quantum communication and knowledge of the electronic properties of QD's interfaced with variety of environments is an important step in this direction. The superconducting–QD tunnel junctions are obtained by coupling a QD with superconducting leads, and provide a way to study

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influence of separation of dot energy levels, coupling parameter between dot states and superconducting leads and Coulomb correlations on quantum transport in these (S-QD-S) Josephson junction.

Recently, the influence of the electron correlations on the Josephson transport across superconductor quantum dot superconductor (S-QD-S) junction has been investigated. One of the interesting issues in quantum transport in nano-scopic materials is the understanding of Josephson electronic current through QD interfaced with superconducting leads[5-8] due to immense technological potential. There have been several attempts to study the electronic transport through correlated QD sandwich between the superconducting leads[9-19]. In these studies, an interplay of the single particle and Josephson Cooper pair tunnelling on super-current across the superconducting quantum dot junction has been analyzed[19-27] . It is pointed out that the Josephson super-current across (S-QD-S) junction depend on the competitive role of the single particle tunnelling and the energy of the dot level with respect to binding energy of the Cooper pairs in superconducting leads and also on the Josephson Cooper pair tunnelling process across (S-QD-S) junction. In these studies a single level quantum dot is considered as a sandwich between superconducting leads having s-wave pairing symmetry[28-30]. At low temperatures, Cooper pairs from one side of superconducting lead get tunnel to other side one -by-one through the discrete energy level of QD without any pair breaking effects as coherence length of superconducting state in conventional superconductors is much larger compared to the size of quantum dots nanostructure. On the other hand, there are studies extended to a double quantum dots in parallel and perpendicular configurations (T-shaped) coupled to superconducting leads [27-28](S-DQD-S junction). The S-DQD-S double quantum dots Josephson junction exhibit rich physics as compared to S-QD-S single quantum dot tunnel junction and provide an opportunity to study theoretically the electronic properties within two impurity extended Anderson model system. Electronic transport through a parallel double quantum dots (DQD) coupled to superconducting leads within numerical renormalization group approach has been analyzed and transmission probability for the system of two coupled identical quantum dots studied [27].

For the case of double quantum dots having inter dot hopping along-with on dot Coulomb repulsion, the conductance through a half-filled double quantum dot thoroughly studied and relation between electronic transport and electron-configuration of double quantum dots have also been analyzed[28] . There are theoretical attempts even for the array of multiple quantum dots placed in parallel configuration between two electron reservoirs in normal state and coupled with the quantum dots through time-dependent tunnelling matrix elements[29]. Recently, the attempts have also been made to analyse two serially aligned quantum dots with large tunnel coupling with the BCS superconductors as leads. It is further pointed out that the electronic hopping between the dots states is strong compared to the coupling between superconductor–dots states [31] and play important role in understanding the electronic structure and nature of Josephson transport in the coupled double quantum dots tunnel junctions [32].

The electronic structure of a double quantum dots device with both the dot coupled to perfect conducting leads strongly influence the local density of states on the dots and there by electronic transport behaviour across the S-DQD-S system. Recently, within Keldysh non-equilibrium Green's function an analysis of local electronic density of states (i.e. electronic spectral density) has also been attempted for multiple dots[33]. Further, it is interesting to pointed out that the tunnelling conductance between a metallic electrode and strongly correlated material can be connected with effective local density of states and in these tunnel junctions defined in terms of integrated single particle spectral function at Fermi level. The influence of coupled quantum dots sand-witched between normal as well as BCS superconducting leads on the electronic transport across such system is not clearly understood so far experimentally as well as theoretically. Therefore, in the present work, we have planned to analyse the spectral density of Cooper pair as a function of various coupling parameters in double T-shaped coupled quantum dots connected to conventional superconducting leads with s-wave pairing symmetry. This analysis is important as electronic spectral density of Cooper pair at Fermi level represents the zero bias conductance across such S-DQD-S tunnel junction and provide an interesting insight into the electronic structure and Josephson transport behaviour in nano- junctions[33-34]. In the next section, we have presented our model Hamiltonian and theoretical formulation for the calculations and analysis of spectral density in S-DQD-S Josephson tunnel junction.

2. Model and Theoretical Formulation:

In order to analyze electronic spectral density of Cooper pairs in superconducting T-shape coupled double quantum dot (S-DQD-S) tunnel junction. We consider Bardeen, Cooper, and Schrieffer (BCS) superconducting leads having order parameter with s-way paring symmetry. Also, it is assumed that-S-DQD-S junction is at low temperatures (ie.T<Tc). Under these conditions, Cooper pairs from one superconducting lead can tunnel through the discrete electronic energy level of double quantum dots to other side one-by-one through the DQD nanostructure. This situation is shown schematically in figure (1), where two single level Quantum Dots coupled with each other in T-shape, while one of the dots is connected to s-wave superconducting leads. The Hamiltonian with double Quantum Dots having s-wave paring symmetry for our (S-DQD-S) Josephson junction can be described as follow:

$$H = H_D + \sum_{L=1,2} \left(H_L + H_T \right) \qquad (1)$$

Where,

$$H_D = \sum_\sigma \left[\left(\varepsilon_1 d^+_{1\sigma} d_{1\sigma} + \varepsilon_2 d^+_{2\sigma} d_{2\sigma} \right) + t_d \left(d^+_{1\sigma} d_{2\sigma} + d^+_{2\sigma} d_{1\sigma} \right) \right] + U_1 n^d_{1\uparrow} n^d_{1\downarrow} + U_2 n^d_{2\uparrow} n^d_{2\downarrow}$$

(1a)

$$H_L = \sum_{k,\eta=1,2} \varepsilon_k c^+_{\eta k\sigma} c_{\eta k\sigma} - \sum_k \left(\Delta_{\eta=1,2} c^+_{\eta k\sigma} c^+_{\eta -k\downarrow} + \Delta^+_{\eta=1,2} c_{\eta -k\downarrow} c_{\eta k\uparrow} \right)$$

(1b)

$$H_T = \sum_{k\sigma} T_{k1} \left(c^+_{1k\sigma} d_{1\sigma} + d^+_{1\sigma} c_{1k\sigma} \right) + \sum_{k\sigma} T_{k2} \left(c^+_{2k\sigma} d_{1\sigma} + d^+_{1\sigma} c_{2k\sigma} \right)$$

(1c)

Where, the H_d (Eq.1a) is the Hamiltonian for correlated double T-shaped QD with the single energy level (ε_1) and (ε_2) respectively. In (Eq.1a) U_1 and U_2 are the on dot Coulomb energy on each dots and initially to avoid complexity, we have considered an uncorrelated case. H_L (Eq.1b) is the BCS effective Hamiltonian to describe superconducting leads. In H_L the second term represents the effective attractive interaction between the electrons of superconducting lead responsible to from bond Cooper pairs and yield a BCS superconducting state described by a gap at the Fermi level, clearly shown in figure (1).The (Eq.1c) represents the possibility of the tunnelling through QD1 and QD2 connected to left and right superconducting leads and QD energy level and vice-versa. The $c_{\eta\sigma k}\left(c^+_{\eta\sigma k}\right)$ represents the annihilation (creation) operators for the superconducting lead $d_\sigma \left(d^+_\sigma\right)$ represents the annihilation (creation) operators for the dot state and we have assumed Fermi energy to be at zero (eV). To study the electronic spectral density of S-DQD-S Josephson junction, we start with the following Green's function: $G_{11} = \langle\langle c_{1k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle$.The equation of motion corresponding to above Green's function [2] can be obtained in following way :

$$\omega \langle\langle c_{1k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = \frac{1}{2\pi} \langle \left[c_{1k\uparrow} \, c^{+}_{1k\uparrow} \right] \rangle + \langle\langle \left[c_{1k\uparrow}, \mathrm{H} \right]; c^{+}_{1k\uparrow} \rangle\rangle \tag{2}$$

Where the Commutator $\left[c_{1k\uparrow}, \mathrm{H} \right]$, is solved in the light of above model Hamiltonian and we get equation of motion with still higher order Green's functions. To linearize higher order Green's function, we follow a BCS- Mean field Approximation [6-7] and get the following coupled Green's functions equation of motion:

$$(\omega - \varepsilon_k) \langle\langle c_{1k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = \frac{1}{2\pi} - \Delta_1 \langle\langle c^{+}_{1-k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle + V_1 \langle\langle d_{1\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{3}$$

$$(\omega + \varepsilon_k) \langle\langle c^{+}_{1-k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = \Delta^{+}_1 \langle\langle c_{1k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_1 \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{4}$$

$$(\omega - \varepsilon) \langle\langle d_{1\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = t_d \langle\langle d_{2\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle + V_1 \langle\langle c_{1k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle + V_2 \langle\langle c_{2k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{5}$$

$$(\omega + \varepsilon) \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = -t_d \langle\langle d^{+}_{2\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_1 \langle\langle c^{+}_{1k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_2 \langle\langle c^{+}_{2k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{6}$$

$$(\omega - \varepsilon) \langle\langle d_{2\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = t_d \langle\langle d_{1\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{7}$$

$$(\omega + \varepsilon) \langle\langle d^{+}_{2\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = -t_d \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{8}$$

$$(\omega + \varepsilon_k) \langle\langle c^{+}_{1k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = \Delta^{+}_1 \langle\langle c_{1-k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_1 \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{9}$$

$$(\omega - \varepsilon_k) \langle\langle c_{2k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = -\Delta_2 \langle\langle c^{+}_{2-k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle + V_2 \langle\langle d_{1\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{10}$$

$$(\omega + \varepsilon_k) \langle\langle c^{+}_{2k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = \Delta^{+}_2 \langle\langle c_{2-k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_2 \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{11}$$

$$(\omega - \varepsilon_k) \langle\langle c_{1-k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle = -\Delta_1 \langle\langle c^{+}_{1k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle + V_1 \langle\langle d_{1\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{12}$$

$$(\omega + \varepsilon_k) \langle\langle c^{+}_{2-k\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle = \Delta^{+}_2 \langle\langle c_{2k\uparrow}; c^{+}_{1k\uparrow} \rangle\rangle - V_1 \langle\langle d^{+}_{1\downarrow}; c^{+}_{1k\uparrow} \rangle\rangle \tag{13}$$

$$(\omega-\varepsilon_k)\langle\langle c_{2-k\uparrow};c^+_{1k\uparrow}\rangle\rangle=-\Delta_2\langle\langle c^+_{2k\downarrow};c^+_{1k\uparrow}\rangle\rangle+V_2\langle\langle d_{1\uparrow};c^+_{1k\uparrow}\rangle\rangle$$
(14)

In above equations $\Delta_1=\sum_k U\langle c_{1-k\downarrow};c_{1k\uparrow}\rangle$ is the superconducting order parameter of the left side ($L=1$) superconductor and $\Delta_2=\sum_k U\langle c_{2-k\downarrow};c_{2k\uparrow}\rangle$ is the superconducting order parameter of the right side ($L=2$) superconductors. Here, we assume that both superconductors are identical and have same superconducting order parameters (i.e. $\Delta_1=\Delta_2=\Delta$ and also $\varepsilon_1=\varepsilon_2=\varepsilon$) and coupling of the dots state with superconducting leads is also symmetrical (i.e. $V_1=V_2=V$). Solving the above coupled equations (3-14), within above conditions, a simple algebra leads to the final expression for the desired Green's function representing superconducting bound Cooper pair correlation as[20-27].

$$\langle\langle c^+_{1-k\downarrow};c^+_{1k\uparrow}\rangle\rangle=\frac{1}{2\pi}\left[\frac{\left\{\begin{array}{l}t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a^*_k\omega^9+b^*_k\omega^7+\\ c^*_k\omega^6+d^*_k\omega^5+e^*_k\omega^4+f^*_k\omega^3+g^*_k\omega^2+h^*_k\omega\end{array}\right\}}{\left\{\begin{array}{l}t_d^2\left(a^1_k\omega^8+b^1_k\omega^6+c^1_k\omega^4+d^1_k\omega^2+e^1_k\omega+f^1_k\right)+a^2_k\omega^{10}+b^2_k\omega^8+\\ c^2_k\omega^6+d^2_k\omega^5+e^2_k\omega^4+f^2_k\omega^3+g^2_k\omega^2+h^2_k\omega+i^2_k\end{array}\right\}}\right] \quad (15)$$

Where,

$a_k=(2+4|\Delta|)$, $\quad b_k=(2\varepsilon^2-4\varepsilon_k^2+|\Delta|\varepsilon_k+V^2+2)$,

$c_k=\left(4|\Delta|\varepsilon^2-4|\Delta|\varepsilon_k^2+V^2\varepsilon_k-V^2\varepsilon+2\varepsilon-4\varepsilon_k\right)$,

$d_k=\left(2\varepsilon_k^4-4\varepsilon^2\varepsilon_k^2+2|\Delta|^2+4|\Delta|\varepsilon^2\varepsilon_k-4|\Delta|\varepsilon_k^3-V^2\varepsilon_k^2-V^2\varepsilon\varepsilon_k+2\varepsilon_k-2\varepsilon\varepsilon_k+2\varepsilon\varepsilon_k^3+3V^2|\Delta|\right)$,

$e_k=\left(4|\Delta|\varepsilon^2\varepsilon_k^2+V^2\varepsilon\varepsilon_k^2-V^2\varepsilon_k^3+2\varepsilon_k^3-2\varepsilon\varepsilon_k^2+V^2|\Delta|\varepsilon-V^2|\Delta|\varepsilon_k\right)$,

$f_k=\left(2\varepsilon^2\varepsilon_k^4+2|\Delta|^2\varepsilon^2-4|\Delta|\varepsilon^2\varepsilon_k^2+V^2\varepsilon\varepsilon_k^3+2\varepsilon\varepsilon_k^3-3V^2|\Delta|\varepsilon\varepsilon_k\right)$,

$a^*_k=1$, $b^*_k=\left(-2\varepsilon^2-2\varepsilon_k^2+2|\Delta|-3V^2\right)$, $c^*_k=\left(-2V^2\varepsilon-V^2\varepsilon_k\right)$,

$d^*_k=\left(\varepsilon^4+\varepsilon_k^4 4\varepsilon^2\varepsilon_k^2+|\Delta|^2-4|\Delta|\varepsilon^2-2|\Delta|\varepsilon_k^2+3V^2\varepsilon^2+3V^2\varepsilon_k^2-3V^2\varepsilon\varepsilon_k-3|\Delta|V^2\right)$,

$e^*_k=\left(-3V^2\varepsilon^3-V^2\varepsilon_k^3-V^2\varepsilon\varepsilon_k^2-V^2\varepsilon^2\varepsilon_k+|\Delta|V^2\varepsilon+|\Delta|V^2\varepsilon_k\right)$,

$f^*_k=\left(-2\varepsilon^2\varepsilon_k^4-2\varepsilon_k^2\varepsilon^4-2|\Delta|^2\varepsilon^2+2|\Delta|\varepsilon^4+4|\Delta|\varepsilon^2\varepsilon_k^2+3V^2\varepsilon\varepsilon_k^3-3V^2\varepsilon^2\varepsilon_k^2+3V^2\varepsilon^3\varepsilon_k+3|\Delta|V^2\varepsilon^2-3|\Delta|V^2\varepsilon\varepsilon_k\right)$,

$g^*_k=\left(V^2\varepsilon^2\varepsilon_k^3+V^2\varepsilon^3\varepsilon_k^2-|\Delta|V^2\varepsilon^3-|\Delta|V^2\varepsilon^2\varepsilon_k\right)$,

$h^*_k=\left(\varepsilon^4\varepsilon_k^4+|\Delta|^2\varepsilon^4-2|\Delta|\varepsilon^4\varepsilon_k^2-3V^2\varepsilon^3\varepsilon_k^3+|\Delta|V^2\varepsilon^3\varepsilon_k+2|\Delta|V^2\varepsilon^3\varepsilon_k\right)$,

$$a_k^1=(-2), \qquad b_k^1=\left(4V^2+6\varepsilon_k^2-2\varepsilon^2-6|\Delta|\right),$$

$$c_k^1=\left(-4V^2\varepsilon\varepsilon_k-8\varepsilon_k^2V^2+6|\Delta|V^2-6\varepsilon_k^4+6\varepsilon^2\varepsilon_k^2-6|\Delta|^2+12\varepsilon_k^2|\Delta|-6|\Delta|\varepsilon^2\right),$$

$$d_k^1=\left(4V^2\varepsilon_k^2+8V^2\varepsilon\varepsilon_k^3+4|\Delta|^2V^2-6|\Delta|V^2\varepsilon_k^2-6|\Delta|V^2\varepsilon\varepsilon_k-6\varepsilon^2\varepsilon_k^4-6|\Delta|^2\varepsilon^2+6|\Delta|^2\varepsilon_k^2-6|\Delta|\varepsilon^4+12|\Delta|\varepsilon^2\varepsilon_k^2\right),$$

$$e_k^1=\left(-2|\Delta|V^2\varepsilon_k^3-2|\Delta|V^2\varepsilon\varepsilon_k^2\right),$$

$$f_k^1=\left(-4\varepsilon\varepsilon_k^5V^2-4|\Delta|^2V^2\varepsilon\varepsilon_k+6|\Delta|V^2\varepsilon\varepsilon_k^3+2\varepsilon^2\varepsilon_k^6+6|\Delta|^2\varepsilon^2\varepsilon_k^2-6|\Delta|\varepsilon^2\varepsilon_k^4-2|\Delta|^3\varepsilon^2\right),$$

$$a_k^2=1, \qquad b_k^2=\left(-4V^2-3\varepsilon_k^2-2\varepsilon^2+3|\Delta|\right),$$

$$c_k^2=\left(8V^2\varepsilon_k^2+4\varepsilon^2V^2-4V^2\varepsilon\varepsilon_k+3\varepsilon_k^4+\varepsilon^4+6\varepsilon^2\varepsilon_k^2+3|\Delta|^2-6|\Delta|\varepsilon_k^2-6|\Delta|\varepsilon^2\right),$$

$$d_k^2=\left(-8|\Delta|V^2\varepsilon_k-4|\Delta|V^2\varepsilon\right),$$

$$e_k^2=\begin{pmatrix}-4\varepsilon_k^4V^2+4V^2\varepsilon^3\varepsilon_k-8V^2\varepsilon^2\varepsilon_k^2+8V^2\varepsilon\varepsilon_k^3-4|\Delta|^2V^2-\varepsilon^2-\varepsilon\varepsilon_k\\ -\varepsilon_k^6-3\varepsilon^4\varepsilon_k^2-6\varepsilon^2\varepsilon_k^4-6|\Delta|^2\varepsilon^2-3|\Delta|^2\varepsilon_k^2+3|\Delta|\varepsilon_k^4+3|\Delta|\varepsilon^4+12|\Delta|\varepsilon^2\varepsilon_k^2+|\Delta|^2\end{pmatrix},$$

$$f_k^2=\left(8|\Delta|V^2\varepsilon_k^3+8|\Delta|V^2\varepsilon^3+4|\Delta|V^2\varepsilon^2\varepsilon_k+4|\Delta|V^2\varepsilon\varepsilon_k^2\right),$$

$$g_k^2=\begin{pmatrix}-8V^2\varepsilon^3\varepsilon_k^3+4V^2\varepsilon^2\varepsilon_k^4-4V^2\varepsilon\varepsilon_k^5+4|\Delta|^2V^2\varepsilon^2-2|\Delta|^2V^2\varepsilon\varepsilon_k-4|\Delta|V^2\varepsilon^2\varepsilon_k^2+4|\Delta|V^2\varepsilon\varepsilon_k^3+\\ \varepsilon^4\varepsilon_k^4+2\varepsilon^2\varepsilon_k^6+3|\Delta|^2\varepsilon^4+6|\Delta|^2\varepsilon^2\varepsilon_k^2-6|\Delta|\varepsilon^4\varepsilon_k^2-6|\Delta|\varepsilon^2\varepsilon_k^4-2|\Delta|^2\varepsilon^2\end{pmatrix},$$

$$h_k^2=\left(-8|\Delta|V^2\varepsilon^3\varepsilon_k^2-4|\Delta|V^2\varepsilon^2\varepsilon_k^3\right),$$

$$i_k^2=\left(4V^2\varepsilon^3\varepsilon_k^5+4|\Delta|^2V^2\varepsilon^3\varepsilon_k-\varepsilon^4\varepsilon_k^6-3|\Delta|^2\varepsilon^4\varepsilon_k^2+3|\Delta|\varepsilon^4\varepsilon_k^4+|\Delta|^2\varepsilon^4\right) \tag{15a}$$

Also,

$\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, and \omega_{10},$ are the poles of Green's function for double coupled quantum dots junction(S-DQD-S). These poles are calculated with the help of numerical computation using MATLAB software, also the spectral weight corresponding to each quasi-particle states (poles) related to Green's function is given by:

$$A_{1k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_1-\omega_2)(\omega_1-\omega_3)(\omega_1-\omega_4)(\omega_1-\omega_5)(\omega_1-\omega_6)(\omega_1-\omega_7)(\omega_1-\omega_8)(\omega_1-\omega_9)(\omega_1-\omega_{10})}\right]$$

$$A_{2k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_2-\omega_1)(\omega_2-\omega_3)(\omega_2-\omega_4)(\omega_2-\omega_5)(\omega_2-\omega_6)(\omega_2-\omega_7)(\omega_2-\omega_8)(\omega_2-\omega_9)(\omega_2-\omega_{10})}\right]$$

$$A_{3k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_3-\omega_1)(\omega_3-\omega_2)(\omega_3-\omega_4)(\omega_3-\omega_5)(\omega_3-\omega_6)(\omega_3-\omega_7)(\omega_3-\omega_8)(\omega_3-\omega_9)(\omega_3-\omega_{10})}\right]$$

$$A_{4k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_4-\omega_1)(\omega_4-\omega_2)(\omega_4-\omega_3)(\omega_4-\omega_5)(\omega_4-\omega_6)(\omega_4-\omega_7)(\omega_4-\omega_8)(\omega_4-\omega_9)(\omega_4-\omega_{10})}\right]$$

$$A_{5k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_5-\omega_1)(\omega_5-\omega_2)(\omega_5-\omega_3)(\omega_5-\omega_4)(\omega_5-\omega_6)(\omega_5-\omega_7)(\omega_5-\omega_8)(\omega_5-\omega_9)(\omega_5-\omega_{10})}\right]$$

$$A_{6k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_6-\omega_1)(\omega_6-\omega_2)(\omega_6-\omega_3)(\omega_6-\omega_4)(\omega_6-\omega_5)(\omega_6-\omega_7)(\omega_6-\omega_8)(\omega_6-\omega_9)(\omega_6-\omega_{10})}\right]$$

$$A_{7k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_7-\omega_1)(\omega_7-\omega_2)(\omega_7-\omega_3)(\omega_7-\omega_4)(\omega_7-\omega_5)(\omega_7-\omega_6)(\omega_7-\omega_8)(\omega_7-\omega_9)(\omega_7-\omega_{10})}\right]$$

$$A_{8k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_8-\omega_1)(\omega_8-\omega_2)(\omega_8-\omega_3)(\omega_8-\omega_4)(\omega_8-\omega_5)(\omega_8-\omega_6)(\omega_8-\omega_7)(\omega_8-\omega_9)(\omega_8-\omega_{10})}\right]$$

$$A_{9k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_9-\omega_1)(\omega_9-\omega_2)(\omega_9-\omega_3)(\omega_9-\omega_4)(\omega_9-\omega_5)(\omega_9-\omega_6)(\omega_9-\omega_7)(\omega_9-\omega_8)(\omega_9-\omega_{10})}\right]$$

$$A_{10k}=\left[\frac{t_d^2\left(a_k\omega^7+b_k\omega^5+c_k\omega^4+d_k\omega^3+e_k\omega^2+f\omega\right)+a_k^*\omega^9+b_k^*\omega^7+c_k^*\omega^6+d_k^*\omega^5+e_k^*\omega^4+f_k^*\omega^3+g_k^*\omega^2+h_k^*\omega}{(\omega_{10}-\omega_1)(\omega_{10}-\omega_2)(\omega_{10}-\omega_3)(\omega_{10}-\omega_4)(\omega_{10}-\omega_5)(\omega_{10}-\omega_6)(\omega_{10}-\omega_7)(\omega_{10}-\omega_8)(\omega_{10}-\omega_9)}\right]$$

(15b)

One can write the above expression (15) in the following standard Green's function from:

$$\left\langle\left\langle c^{+}_{1-k\downarrow};c^{+}_{1k\uparrow}\right\rangle\right\rangle=\frac{1}{2\pi}\sum_{i=1}\frac{A_{ik}}{(\omega-\omega_i)} \tag{16}$$

Where, ω_i's are the poles (i=1 to 10) of Green's functions and represent the quasi-particle energy branches of electronic structure in S-DQD-S junction. The Green's function is related to the correlation function and imaginary part of Green's function provide spectral density of Cooper pair as above equation (16) represents correlation corresponding to bound Cooper pairs in S-DQD-S junction. The spectral density of Cooper pairs can be calculated from the above Green's function $\left\langle\left\langle c^{+}_{1-k\downarrow};c^{+}_{1k\uparrow}\right\rangle\right\rangle$ representing the probability amplitude of bound paired state of electron with momentum k, (spin up↑) and –k with (spin down↓) by using the relationship :

$$A_k(\omega)=-\frac{1}{\pi}\mathrm{Im}\left\langle\left\langle c^{+}_{1-k\downarrow};c^{+}_{1k\uparrow}\right\rangle\right\rangle \tag{17}$$

Where,(Im) stands for imaginary part of Green's function given by equation (16) and involves δ-functions. In order to solve these (delta) functions, we have considered lorentzian type of broadening of the Cooper pair spectral density peaks by using the relationship between δ-function and lorentzian function as follows:

$$\delta(\omega-\omega_i)=\frac{1}{\pi}\lim_{\Gamma\to 0}\frac{\Gamma}{\Gamma^2+(\omega-\omega_i)^2} \tag{18}$$

The broadening factor Γ is taken to be independent of ω and k to avoid complicity at this stage. There for, finally, we get expression of spectral density of Cooper pair for S-DQD-S Josephson junction as follow;

$$A_{k(\omega)}=\frac{\Gamma}{(\pi)^2}\sum_{i=1,}^{10}\frac{A_{ik}}{\left((\Gamma)^2+(\omega-\omega)^2\right)} \tag{19}$$

We have analyzed above electronic spectral density of Cooper pairs as a function of various parameters of model Hamiltonian (equation1) performing numerical computation and employing METLAB software for this purpose. The spectral density of Cooper pairs in S-DQD-S junction predict the nature of Josephson transport in such tunnel junction and clues for maximum Josephson super-current. In the preceding section we have discussed our numerical results.

3. Results and Discussions:

The theoretical expression of spectral density of Cooper pairs (equation-19) in S-DQD-S Josephson junction has been obtained within BCS mean field approximation and Green's function approach for a renormalized two impurity Anderson model. In order to analyze the spectral density A(ω) Vs energy (ω) (initially, shown in figure 2) for different values of the coupling between the two quantum dots states (td), we have performed numerical computation, keeping other parameters as fixed (€=0.03eV, v=0.05eV, €k=0.04eV, Δ=0.005eV, Γ=0.05eV, where Γ is the broadening factor) .When coupling between the dots states (td) increases the spectral density of Cooper pair show(figure 2) multiple peak structure with a prominent spectral peak above and below the Fermi level. These peaks further get sharpened with increase in coupling between the dots. This indicates that the electronic Cooper pair transport across S-DQD-S Josephson junction get tuned with the enhancement of inter-dot coupling as these spectral peaks represent nature of zero bias conductance: a manifestation of Josephson tunnelling across the S-DQD-S junction.

The influence of coupling between the superconducting leads and dot energy level (V) is shown in figure (3), keeping other parameters fixed (td=1.4eV, €=0.03eV, €k=0.04eV, Δ=0.001eV, Γ=0.05eV). One can reveal from figure (3) that the spectral density strongly depends on V, and on increasing coupling between superconducting leads and dot there is a strong suppression of spectral peak below and above the Fermi level. Further, it is important to pointed out here, that below the Fermi level spectral peak position remain unaltered with V though the spectral weight get suppressed. On the other hand, on increasing V (0.5eV, 1.0eV, 1.5eV) the spectral density of Cooper pair get suppressed above the Fermi level along-with the shift in spectral peak away from Fermi level. This implies that probability of Cooper pair transmission across such S-DQD-S junction get suppressed, when single particle coupling between leads (superconductors) and dots is strong and dictate the Josephson Cooper pair tunnelling.

The spectral density of Cooper pairs as a function of dot energy level (€) is shown in (figure 4) while keeping other parameters fixed (td=1.1eV, v=0.5eV, €k=0.04eV, Δ=0.001eV, Γ=0.05). It is important to note from (figure 4) that on changing the position of dot energy level (€) with respect to Fermi level ($€_F$=0), the spectral weight get transferred from the prominent peak above the Fermi level and appear in the form of a spectral peak below Fermi energy without any change in the location of the spectral peaks.Thus position of energy level (€) of the dots with respect to Fermi level bring the spectral weight close to the Fermi level and can be used as a tunable parameter to control Josephson Cooper pair tunnelling in S-DQD-S junctions.

The influence of superconducting order parameter (Δ) i.e. binding energy of Cooper pairs in leads on the spectral density is shown in figure(5), keeping other parameters fixed (td=1.5eV, v=0.05eV, €k=0.07eV, €=0.02eV, Γ=0.05 and Δ=0.003eV). It is interesting to note that on increasing the binding energy of Cooper pairs (Δ=0.01eV to Δ=0.02eV) two sharp spectral peak around the Fermi Level got visible, while the prominent peak away from Fermi level show a small transfer of spectral weight without any change in position of peak due to enhancement of Josephson Cooper pair tunnelling across S-DQD-S tunnel junction. Thus superconducting order parameter play important role in stabilizing Josephson transport across S-DQD-S junction as the spectral density of Cooper pairs show evolution of spectral weight around the Fermi level in the form of two well separated spectral peaks: a signature of enhanced Josephson effect. On the other hand, the coupling between the superconducting leads and quantum dots energy level (V) suppress the Cooper pairs tunnelling across such S-DQD-S hybrid tunnel junction.

Therefore, we have concluded that in T-shaped coupled double quantum dots-superconductor (S-DQD-S) tunnel junction the spectral density of Cooper pairs strongly influenced by the dots energy levels, coupling parameter of the junction as well as nature and magnitude of superconducting order parameter of the leads. The spectral density of Cooper pairs in such tunnel junction in the nature of zero bias conductance and display a controllable Josephson Cooper pair tunnelling across the junction. It will be interesting to analyze Josephson super-current across such S-DQD-S tunnel junction theoretically and generalise these results for the case of series coupled double and Triplet quantum dots- superconductors Josephson junction to provide clues for the experimental studies to lead maximum Josephson current and harness potential applications of the same in electronic devices.

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Figure Captions:

Figure (1): Schematic presentation of the superconducting T-shape coupled double QD Josephson Junction(S-DQD-S).

Figurer (2): The variation of spectral density of Cooper pairs (A(ω) vs ω) with coupling between the dot states (td) in figure(2), keeping other parameters as fixed (€=0.03eV, v=0.05eV, €k=0.04eV, Δ=0.005eV, Γ=0.05eV, where Γ is the broadening factor).

Figurer(3): The influence of coupling between superconducting lead and dot energy level (v) on spectral density of cooper Pairs is shown in (figure 3) keeping other parameters fixed (td=1.4eV, €=0.03eV, €k=0.04eV, Δ=0.001eV, Γ=0.05eV).

Figurer(4): The spectral density of Cooper pairs as a function of dot energy level (€) with respect to Fermi level is shown in (figure 4) while keeping other parameters fixed (td=1.1eV, v=0.5eV, €k=0.04eV, Δ=0.001eV, Γ=0.05eV).

Figurer(5): The influence of superconductivity order parameter (Δ) ie. binding energy of Cooper pair on spectral density of Cooper pair and other parameters fixed is shown in (figure 5) (td=1.5eV, v=0.05eV,
k=0.07eV, €=0.02eV, Γ=0.05).

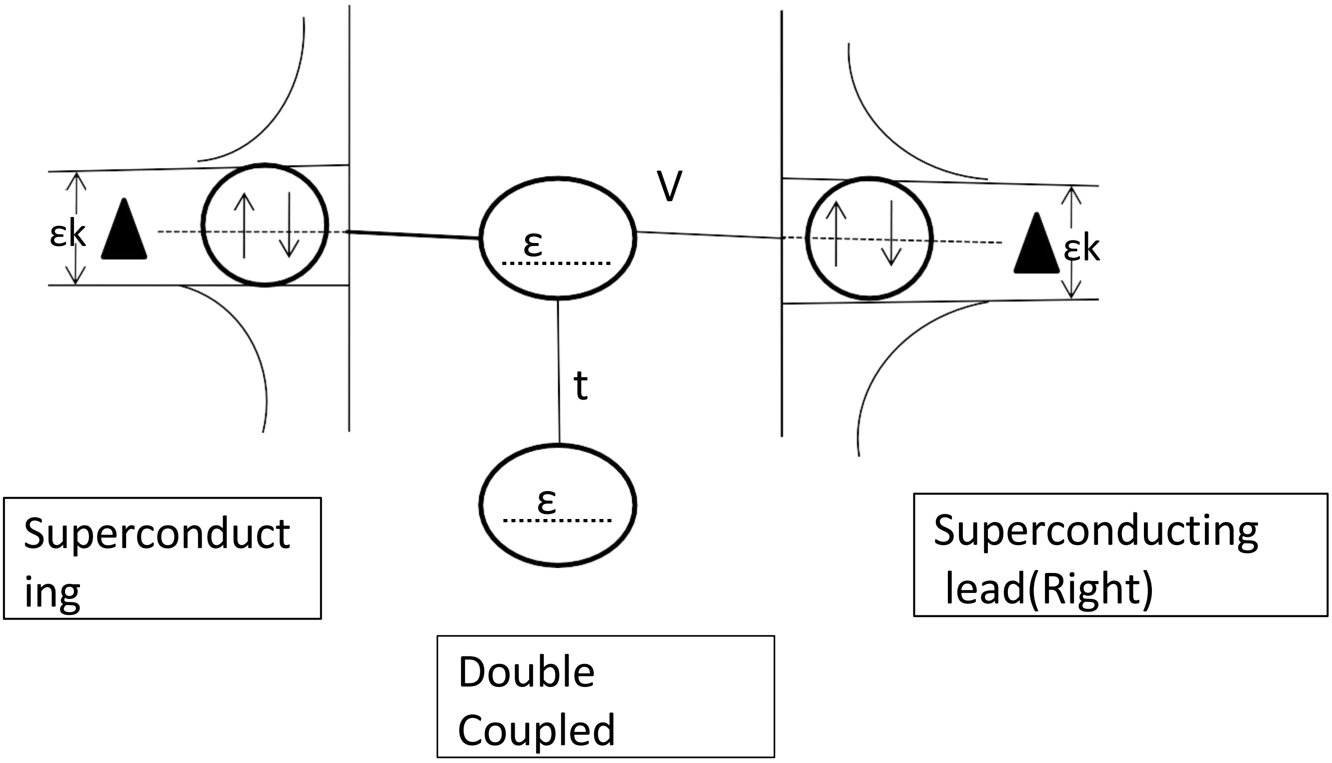


Figure (1).

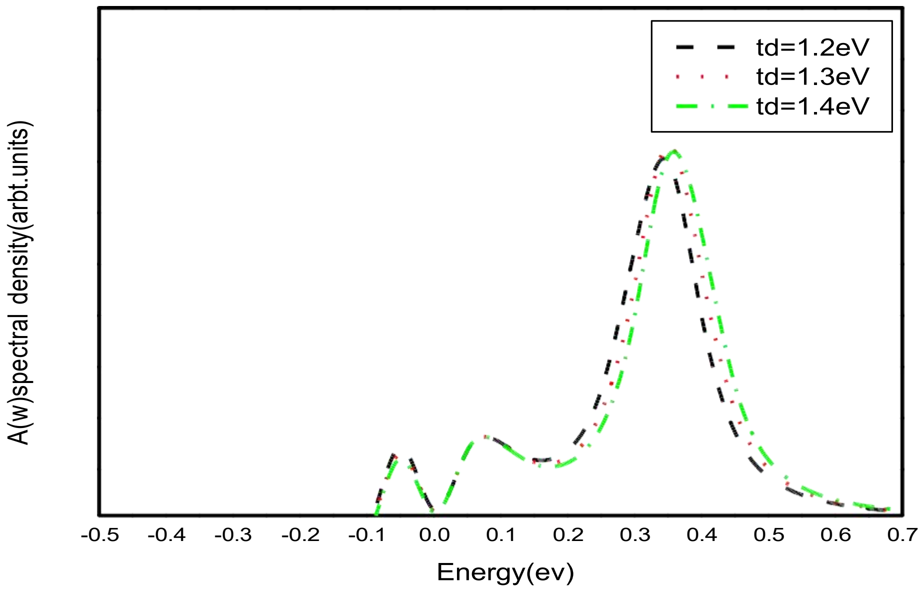
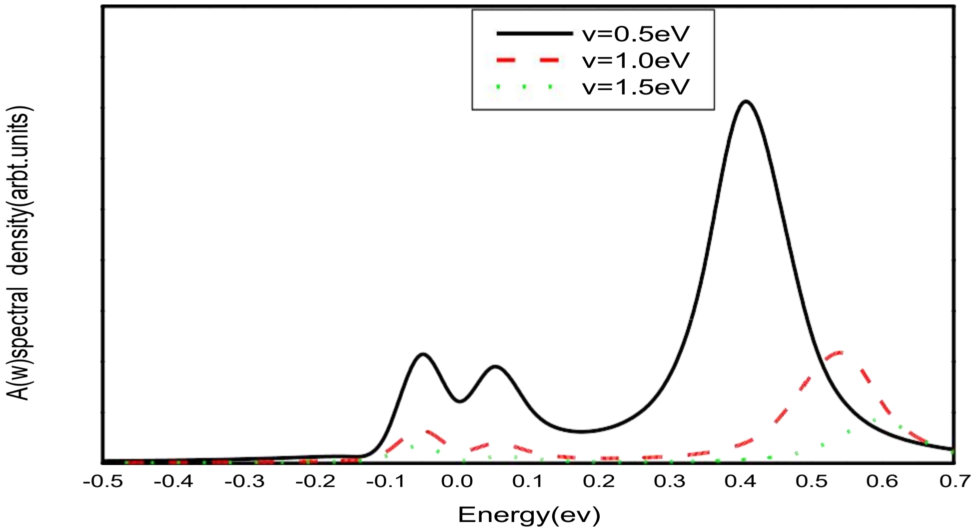
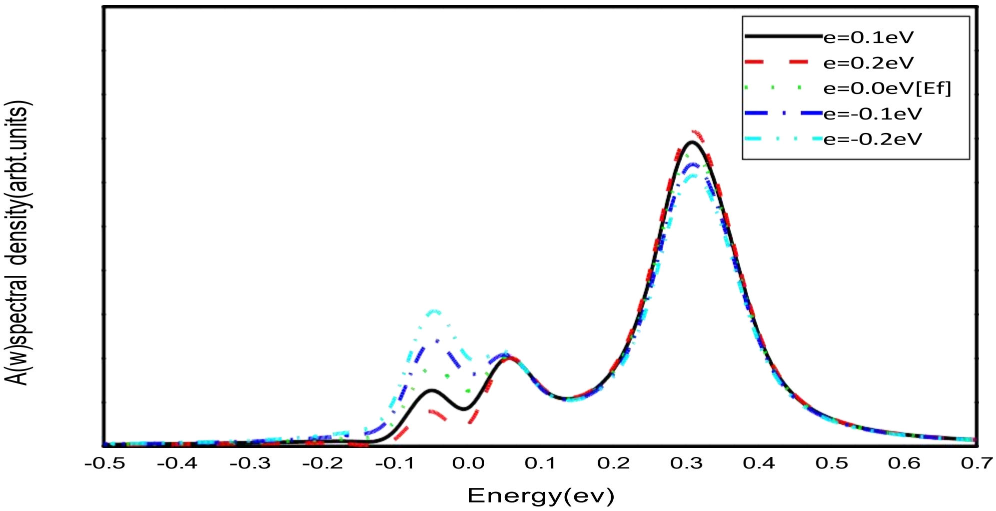


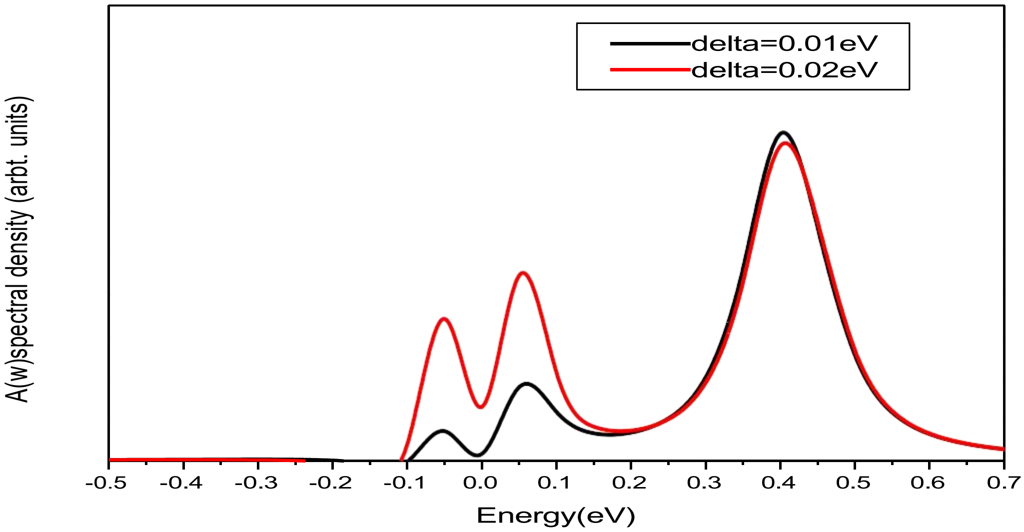
Figure 2



Figurer 3



Figure; 4.


delta=0.01eV
delta=0.02eV
A(w)spectral density (arbt. units)
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Energy(eV)

Figure; 5.